Filters in the z-Plane

Here we look at coming up with a digital filter, G(z), which is equivalent to an anlog filter, G(s). Equivalent meaning

- They both have the same step response, or
- They both have the same frequency response.

For example, the following filters have the same step response. They are equivalent

- One is analog (s-plane)
- One is digital (z-plane)

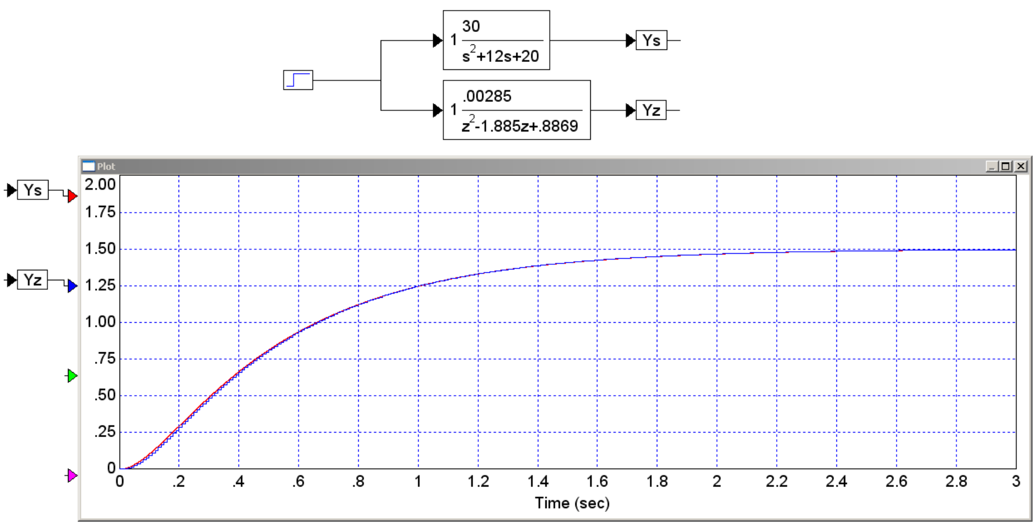

Figure 1: Two filters, one analog and one digital, have the same step response. Hence, the filters are equivalent

The basic assumption behind LaPlace transforms is that all functions are in the form of

$$y(t) = e^{st}$$

This assumptions turns differential equations into algebraic equations in 's' where sY means 'the derivative of Y'.

$$\frac{dy}{dt} = s \cdot e^{st} = sy$$

A transfer function, such as

$$Y = \left(\frac{2s+3}{s^2+2s+10}\right)X$$

thus represents the differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 10y = 2\frac{dx}{dt} + 3x$$

The basic assumption behind z-transforms is that all functions are in the form of

$$y(k) = z^k$$

This assumption turns difference equations into algebraic equations in 'z' where zY means 'the next value of Y'

$$y(k+1) = z^{k+1} = z \cdot z^k = zy(k)$$

A transfer function, such as

$$Y = \left(\frac{2z+3}{z^2+2z+10}\right)X$$

this represents a difference equation

y(k+2) + 2y(k+1) + 10y(k) = 2x(k+1) + 3x(k)

The relationship between the s-plane and z-plane is as follows. Assume that time is sampled at a sampling rate of T seconds. Then

$$t = kT$$

where k is the sample number. Plugging this into the LaPlace assumption

$$y(kT) = e^{skT}$$

$$y(k) = (e^{sT})^k$$

or

$$y(k) = z^k$$

Hence, the transformation between the s-plane and the z-plane is

$$\boxed{z = e^{sT}}$$

Gain for a Sinusoidal Input

If filter, G(s), has a sinusoidal input, you can find the gain at that frequency by substituting

$$s \rightarrow j\omega$$

The result is a comples number where

- the amplitude is how much the signal is scaled, and
- the angle is the phase shift

For example, find y(t) assuming

$$Y = \left(\frac{2}{s+3}\right) X$$

$$x(t) = 4\cos(5t)$$

Solution: Determine the gain at s = j5

$$\left(\frac{2}{s+3}\right)_{s=j5} = 0.343\angle -59^0$$

$$y(t) = (0.343\angle -59^0) \cdot 4\cos(5t)$$

so

$$\boxed{y(t) = 1.372\cos(5t - 59^0)}$$

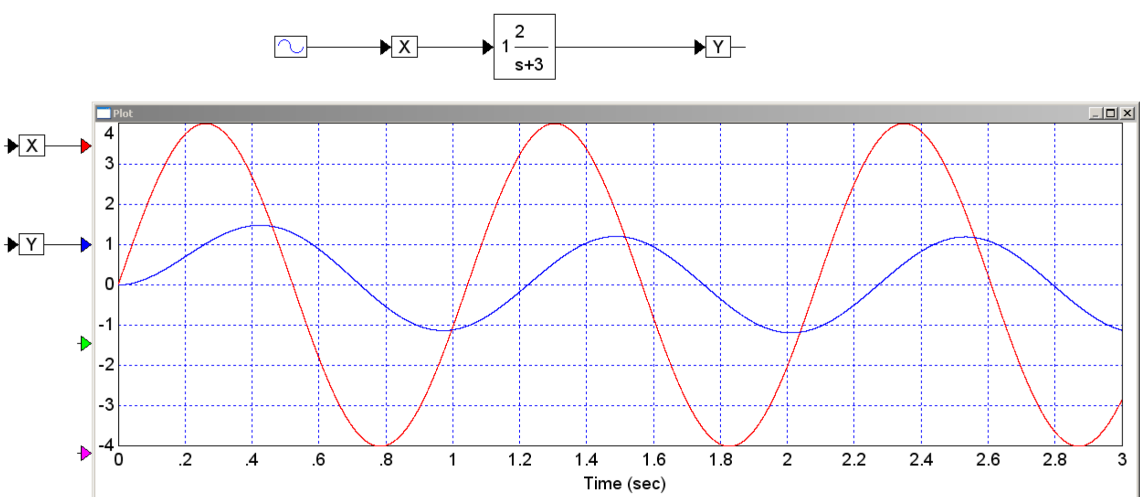

Figure 2: Time vs. voltage plot for a continuous-time filter.
Note that the output is a sine wave with a different amplitude and phase shift suggesting phasor analysis

If a filter, G(z), has a sinusoidal input, you can find the gain at that frequency by substituting

$$z = e^{sT} = e^{j\omega T}$$

For example, find y(t) assuming a sampling rate of 0.01 second (T = 0.01)

$$Y = \left(\frac{0.2}{z-0.9}\right)X$$

$$x(t) = 4\cos(5t)$$

Solution: Determine the gain at $z = e^{j5T}$

$$\left(\frac{0.2}{z-0.9}\right)_{z=e^{j0.05}} = 1.8071\angle-26.84^0$$

so

$$y(t) = (1.8071\angle-26.84^0)\cdot 4\cos(5t)$$

$$y(t) = 7.2282\cos(5t-26.84^0)$$

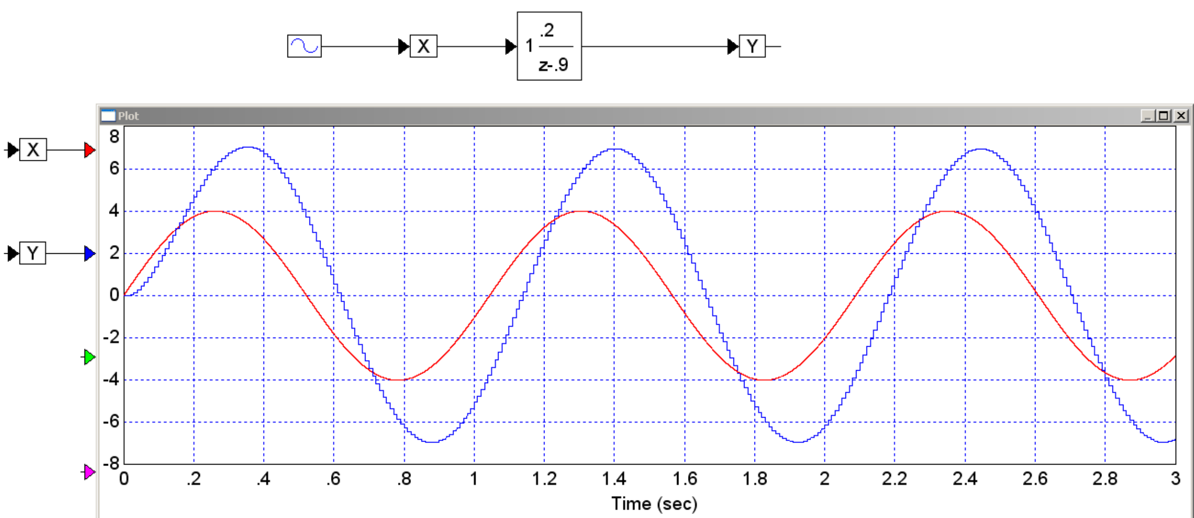

Figure 3: Time vs. voltage plot for a discrete-time filter.
Note that the output is a sine wave with a different amplitude and phase shift suggesting phasor analysis

Note that if you change the sampling rate, you change z as well as the gain. Digital filters require a fixed known sampling rate.

Converting G(s) to G(z)

You can convert a filter from the s-plane to the z-plane using the relationship $z = e^{sT}$. Given G(s)

- Convert the zeros to the z-plane as $z = e^{sT}$
- Convert the poles to the z-plane as $z = e^{sT}$
- Add a gain to match the gain at one frequency (typically DC)

Example 1: Find a digital filter with approximately the same frequency response as

$$G(s) = \left(\frac{30}{(s+2)(s+10)}\right)$$

Assume a sampling rate of 10ms (T = 0.01)

Solution: Convert the poles to the z-plane

s = -2

$$z = e^{-2T} = 0.9802$$

s = -10

$$z = e^{-10T} = 0.9048$$

so

$$G(z) = \left(\frac{k}{(z-0.9802)(z-0.9048)}\right)$$

To find k, match the gain at DC

$$\left(\frac{30}{(s+2)(s+10)}\right)_{s=0} = 1.5$$

so

$$\left(\frac{k}{(z-0.9802)(z-0.9048)}\right)_{z=1} = 1.5$$

k = 0.0028

and

$$G(z) = \left(\frac{0.0028}{(z-0.0802)(z-0.9048)}\right)$$

Hence, for T = 0.01

$$\boxed{\left(\frac{30}{(s+2)(s+10)}\right) \approx \left(\frac{0.0028}{(z-0.0802)(z-0.9048)}\right)}$$

Checking: if the two systems are equivalent, they should have the same step response

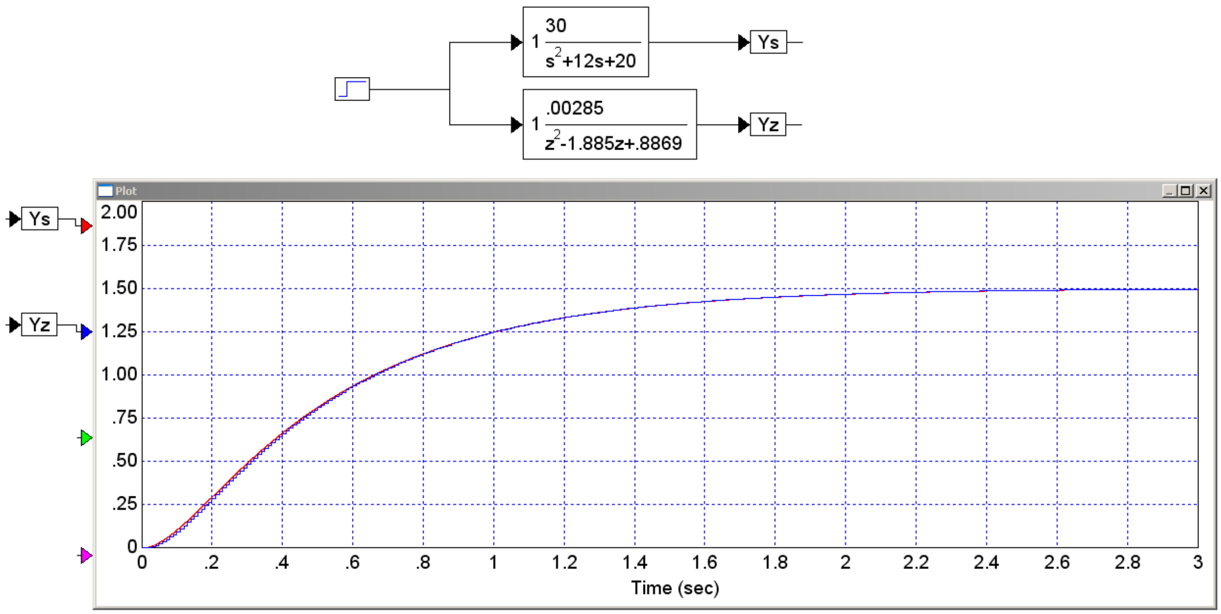

Figure 4: Step response of digital filter, G(z), and analog filter, G(s).
Note that the step response is the same - hence the filters are equivalent.:

```
w = [0:0.01:10]';
j = sqrt(-1);
s = j*w;
Gs = 30 . / ( (s+2).*(s+10) );
T = 0.01;
z = exp(s*T);
Gz = 0.00285 ./ ( (z-0.9801).*(z-0.9048) );
plot(w,abs(Gs),w,abs(Gz));
xlabel('rad/sec');
ylabel('Gain');
```

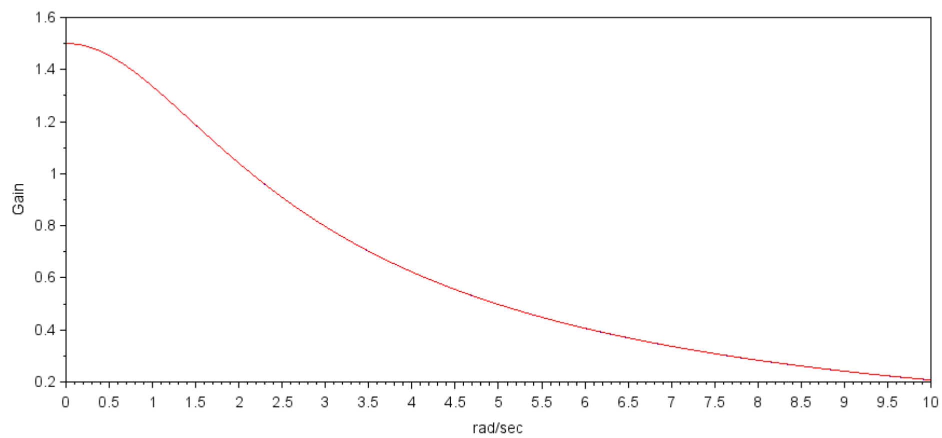

Figure 5: Gain vs. frequency for the G(s) (blue) and G(z) (red) (too close to tell apart)

Programming G(z)

Given a filter G(z), a program to impliment it is fairly straight forward:

- First find the difference equation
- Solve for the highest value of y(k+n)
- Time shift it (change of variable) so that the filter is causal

Example: Write a program to impliment

$$Y = \left(\frac{0.2(z-0.9)}{z^3-1.3z^2+1.6z+0.6}\right)X$$

Cross multiply

$$(z^3 - 1.3z^2 + 1.6z + 0.6)Y = 0.2(z-0.9)X$$

Write the difference equation

```
y(k+3) - 1.3y(k+2) + 1.6y(k+1) + 0.6y(k) = 0.2( x(k+1) - 0.9x(k) )
```

Solve for the highest value of y(k+2)

```
y(k+3) =  1.3y(k+2) - 1.6y(k+1) - 0.6y(k) = 0.2( x(k+1) - 0.9x(k) )
```

Time shift (change of variable: k+3 = k')

```
y(k') =  1.3y(k'-1) - 1.6y(k'-2) - 0.6y(k'-3) + 0.2( x(k'-2) - 0.9x(k'-3) )
```

Write a program. For this program, you need to remember the last three inputs and outputs

```
while(1) {
   x3 = x2;
   x2 = x1;
   x1 = x0;
   x0 = A2D_Read(0);

   y3 = y2;
   y2 = y1;
   y1 = y0;
   y0 = 1.3*y1 - 1.6(y2 - 0.6*y3 + 0.2*( x2 - 0.9*x3 );

   D2A(y0);

   Wait_T();        // set the sampling rate, presumably with interrupts
   }
```

Note

- If you want to change the filter, you just change one line of code
- If you want comples poles or zeros, just choose coefficients that have complex roots

Digital filters are much much easier to impliment than analog filters.