
Uniform Distribution

ECE 341: Random Processes

Lecture #14

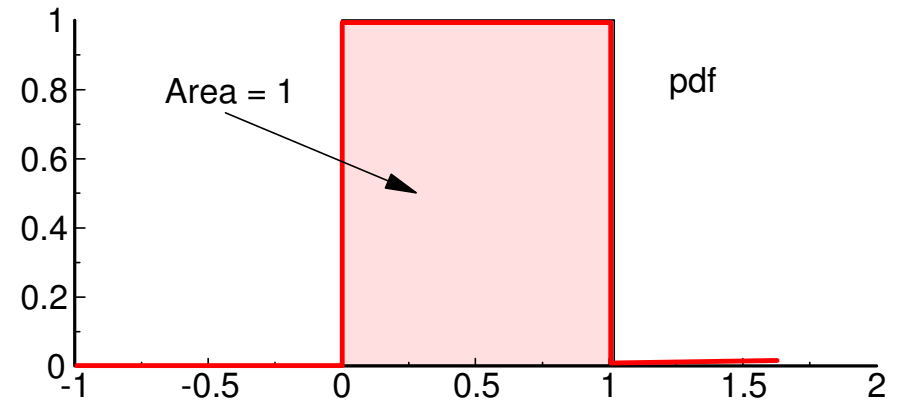
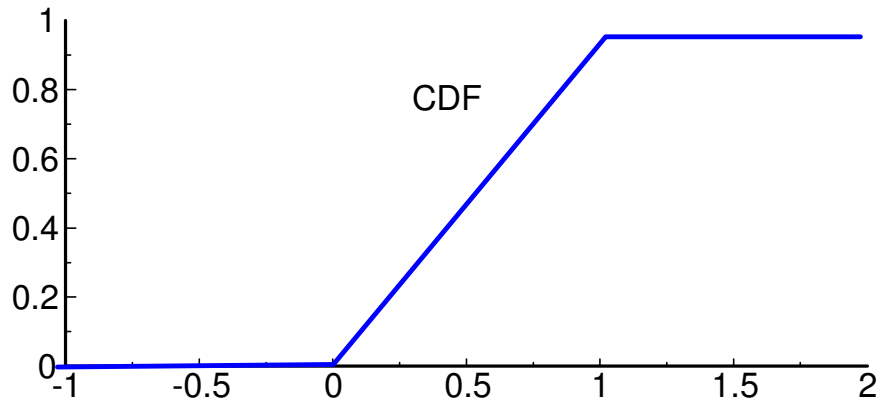
note: All lecture notes, homework sets, and solutions are posted on www.BisonAcademy.com

Uniform Distribution

A uniform distribution is one where the probability density function (pdf) is

- constant over a range (a, b) , and
- zero otherwise.

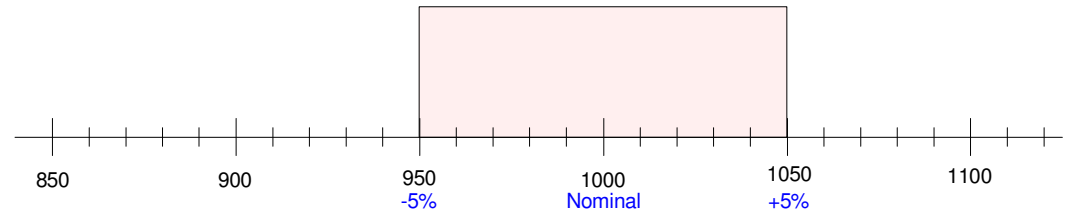
Example: Uniform distribution over the interval of $(0,1)$:



Examples:

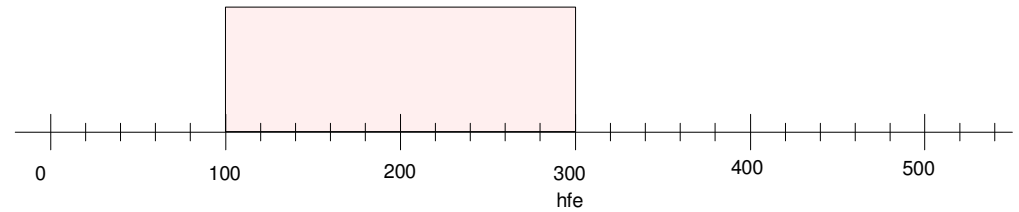
A 1k resistor

- 5% tolerance
- $R = 1k \pm 5\%$



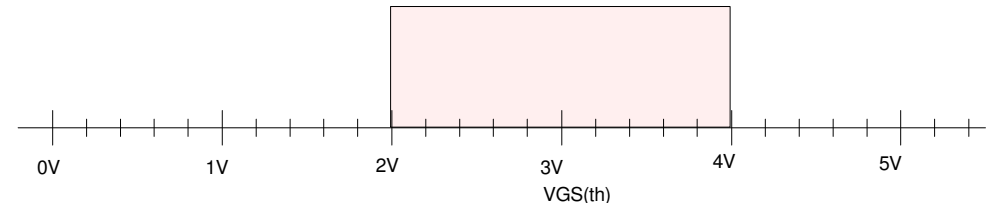
BJT Transistor

- $\min(h_{fe}) = 100$
- $\max(h_{fe}) = 300$



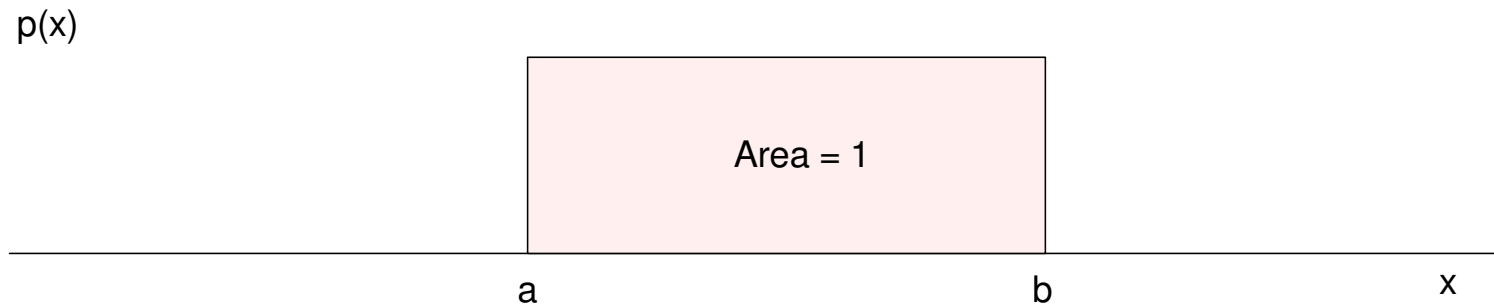
MOSFET

- $\min(V_{GS(th)}) = 2V$
- $\max(V_{GS(th)}) = 4V$



Properties of Uniform Distributions:

Assume a uniform distribution over the interval (a, b)



Since the area must be 1.000, the height must be

$$\text{Area} = \text{width} * \text{height} = 1$$

$$p(x) = \left(\frac{1}{b-a} \right) \quad a < x < b$$

Mean: The mean of the function (almost by inspection) is

$$\bar{x} = \left(\frac{a+b}{2} \right)$$

Variance:

$$\sigma^2 = \int_a^b p(x) (x - \mu)^2 dx$$

$$\sigma^2 = \left(\frac{(b-a)^2}{12} \right)$$

Moment Generating Function

$$\psi(s) = \left(\frac{1}{(b-a)s} \right) (e^{-as} - e^{-bs})$$

Combinations of Uniform Distributions

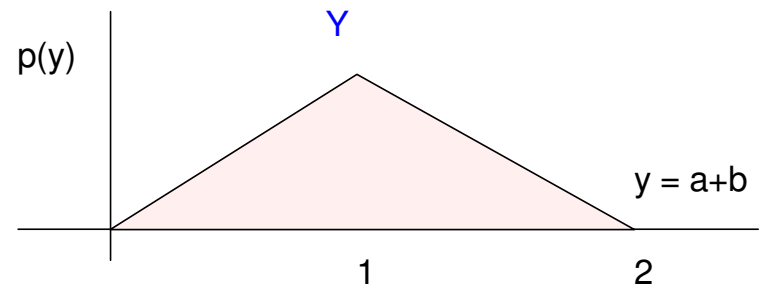
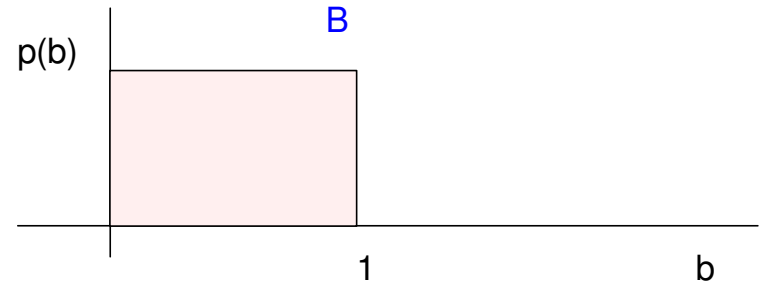
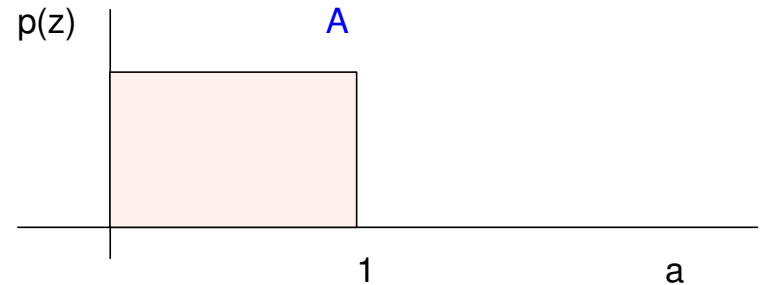
- Net resistance of two resistors in series

Adding two uniform distributions, the results in

- The convolution of their pdf's, or
- The product of their moment generating functions.

Example:

- Assume A and B are uniform distributions over the interval (0, 1).
- Find the pdf of the sum $Y = A + B$.



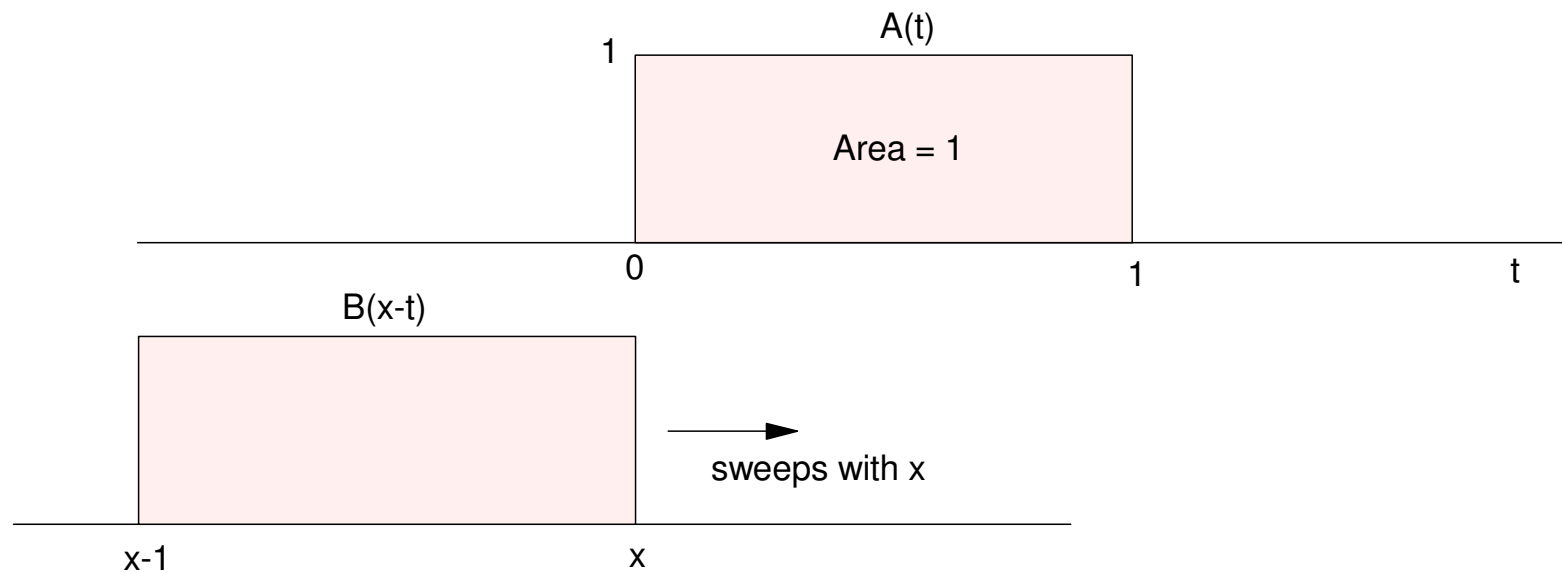
Solution using convolution:

$$A(x) = u(x) - u(x - 1)$$

$$B(x) = u(x) - u(x - 1)$$

$$Y(x) = A(x) * B(x)$$

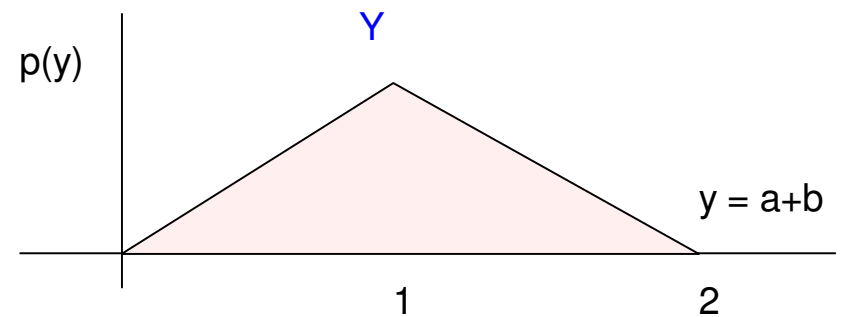
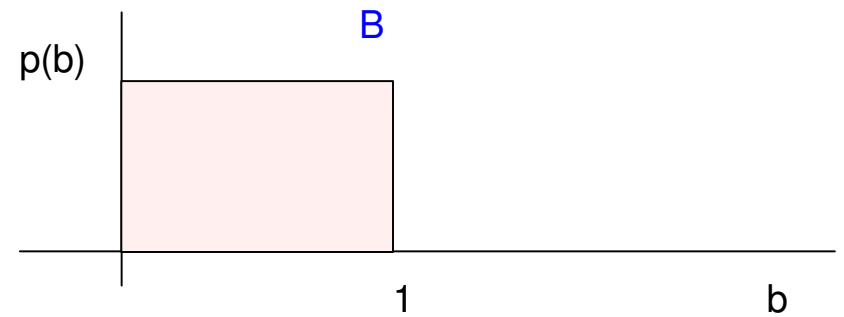
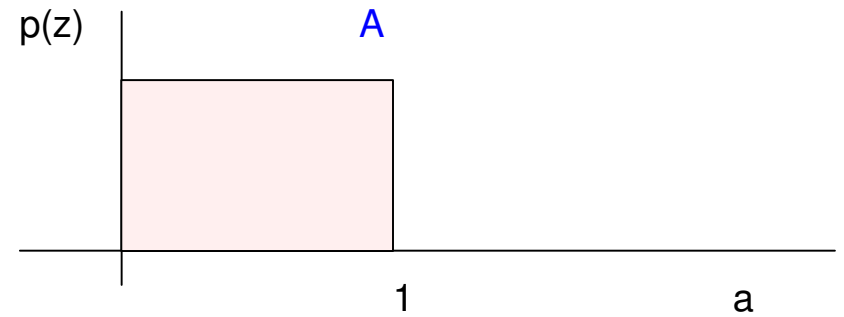
$$Y(x) = \int_{-\infty}^{\infty} A(t)B(x - t)dt$$



From the graph, this integral is

$Y(x) =$

- 0 $x < 0$
- x $0 < x < 1$
- $1 - x$ $1 < x < 2$
- 0 $2 < x$



Solution using Calculus:

$$Y(x) = \int_{-\infty}^{\infty} A(t) B(x-t) dt$$

$$Y(x) = \int_{-\infty}^{\infty} (u(t) - u(t-1)) (u(x-t) - u(x-t-1)) dt$$

Multiply out giving four terms

$$\begin{aligned} &= \int_{-\infty}^{\infty} u(t)u(x-t) dt + \int_{-\infty}^{\infty} -u(t)u(x-t-1) dt \\ &+ \int_{-\infty}^{\infty} -u(t-1)u(x-t) dt + \int_{-\infty}^{\infty} u(t-1) u(x-t-1) dt \end{aligned}$$

$$\begin{aligned} &= \int_0^x 1 dt - \int_0^{x-1} 1 dt - \int_1^x 1 dt + \int_1^{x-1} 1 dt \\ &= xu(x) - (x-1)u(x-1) - (x-1)u(x-1) + (x-2)u(x-2) \\ &= xu(x) - 2(x-1)u(x-1) + (x-2)u(x-2) \end{aligned}$$

Solve using moment generating functions

$$A(s) = \left(\frac{1}{s}\right)(1 - e^{-s})$$

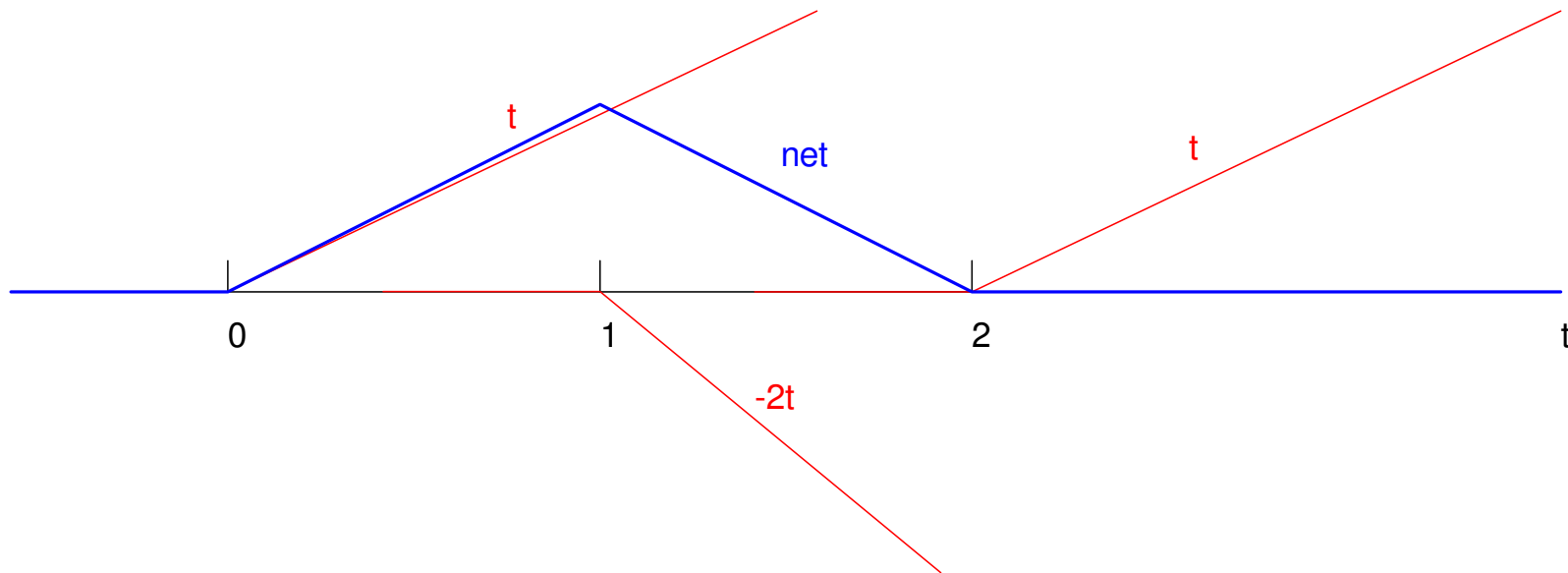
$$B(s) = \left(\frac{1}{s}\right)(1 - e^{-s})$$

$$Y(s) = A(s)B(s)$$

$$Y(s) = \left(\frac{1}{s^2}\right)(e^{-2s} - 2e^{-s} + 1)$$

Take the inverse LaPlace transform

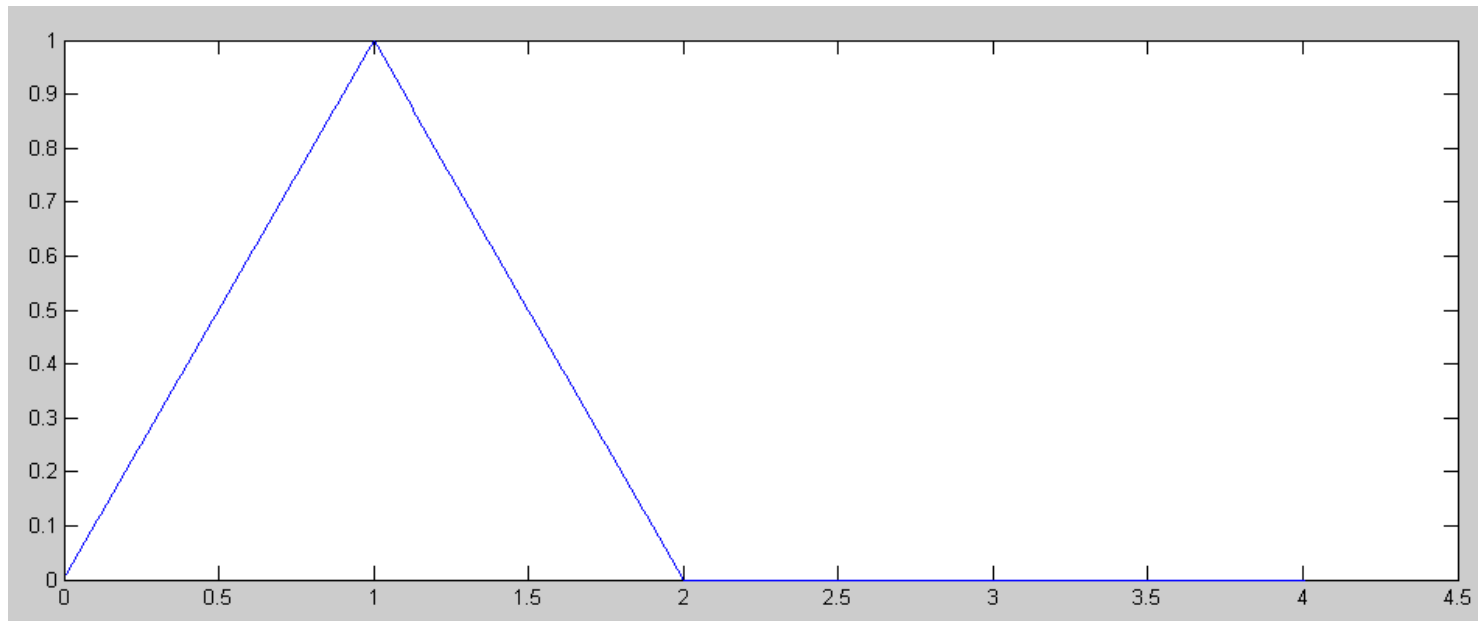
$$y(x) = (x - 2) u(x - 2) - 2(x - 1) u(x - 1) + x u(x)$$



Solve in Matlab

- Approximate a uniform distribution with 100 points over the interval (0, 1)

```
dx = 0.01;  
x = [0:dx:2]';  
A = 1*(x<1);  
B = 1*(x<1);  
Y = conv(A, B) * dx;  
plot([1:length(Y)]*dx, Y)
```

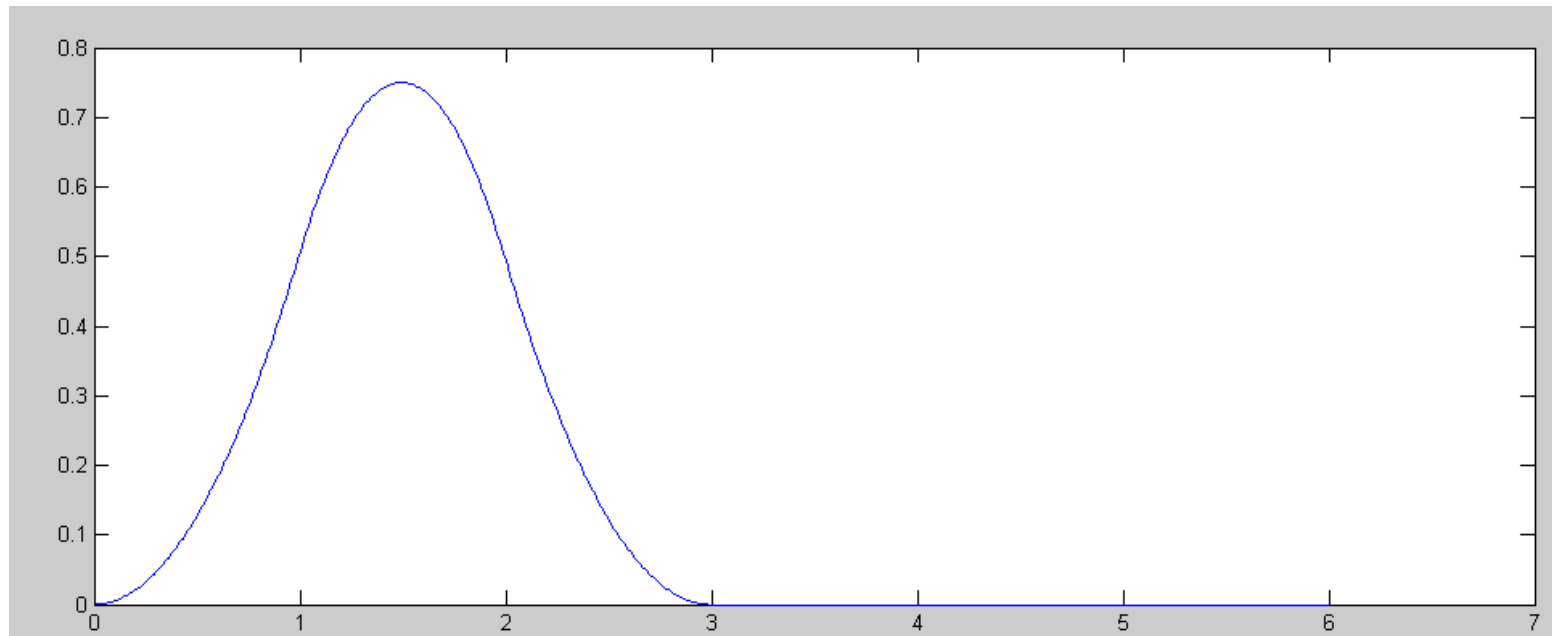
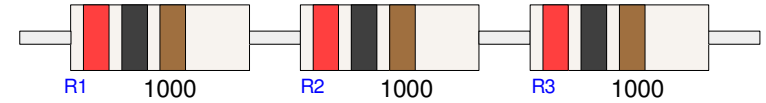


Example 2: Three uniform distributions

- Example: Three resistors in series

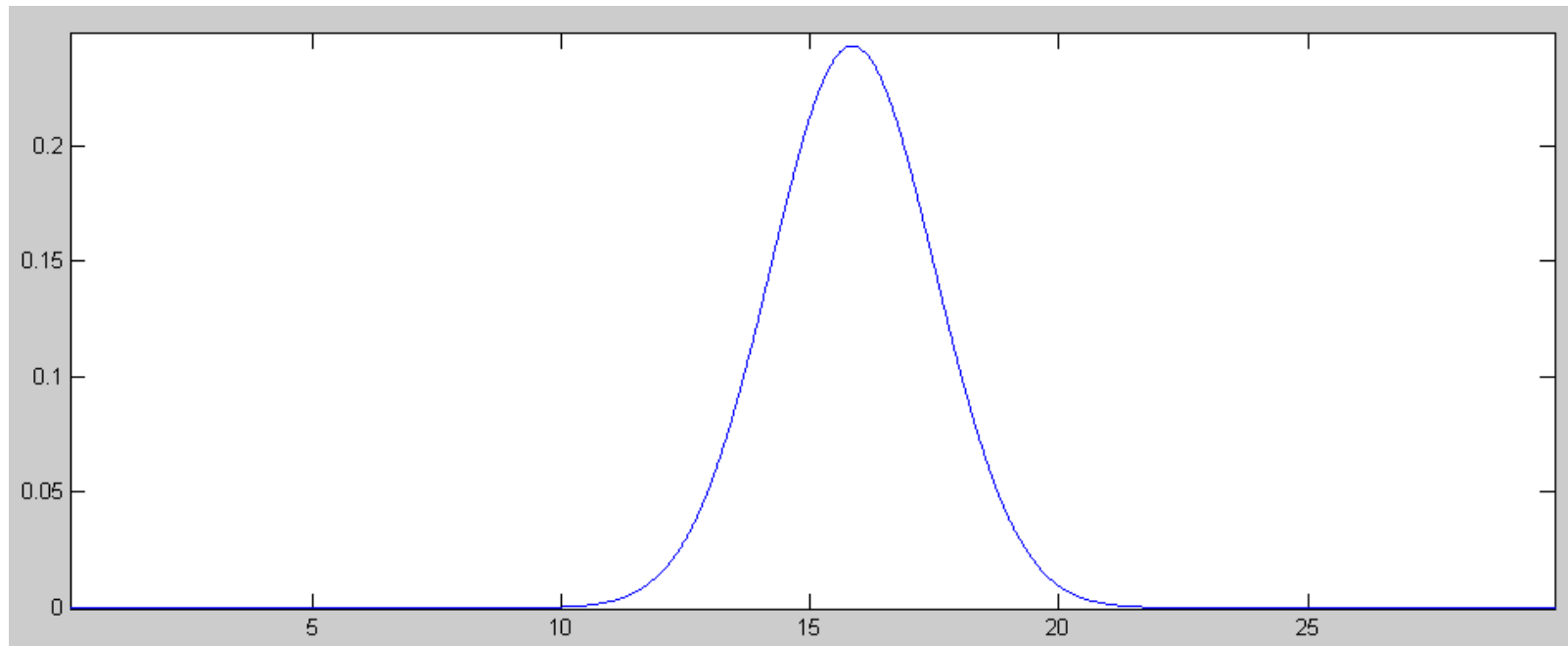
```
x = [0:dx:2]';  
A = 1*(x<1);  
B = 1*(x<1);  
C = 1*(x<1);
```

```
Y = conv(A, B) * dx;  
Y = conv(Y, C) * dx;  
plot([1:length(Y)]*dx, Y)
```



Example 3: Find the pdf for the sum of 32 uniform distributions:

```
x = [0:dx:2]';  
A = 1*(x<1);  
Y2 = conv(A, A) * dx;  
Y4 = conv(Y2, Y2) * dx;  
Y8 = conv(Y4, Y4) * dx;  
Y16 = conv(Y8, Y8) * dx;  
Y32 = conv(Y16, Y16) * dx;  
plot([1:length(Y32)]*dx, Y32)
```



Uniform Distribution in Circuit Analysis:

Find the pdf for the net resistance

- Each resistor has 5% tolerance

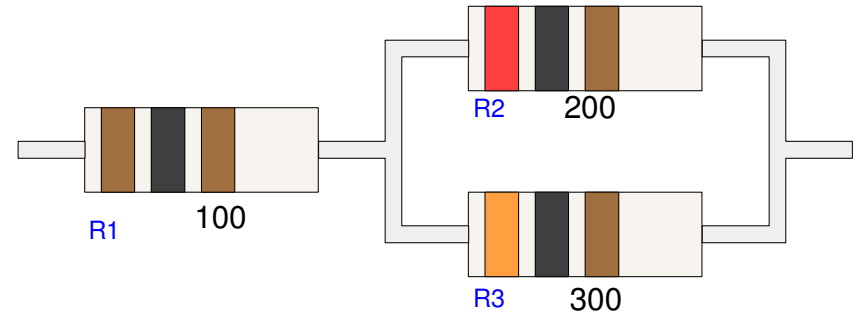
From Circuits I

$$R = R_1 + R_2 || R_3$$

$$R = R_1 + \left(\frac{1}{R_2} + \frac{1}{R_3} \right)^{-1}$$

Problem:

- How do you combine uniform pdf's when you multiply?
- When you divide?



Monte-Carlo to the Rescue!

Monte-Carlo always works

cdf:

- Generate random values for R1, R2, R3
 - Uniform distribution
 - +/- 5%
- Solve for Rnet
- Repeat N tims

When done,

- Sort the data
- Plot the sorted data

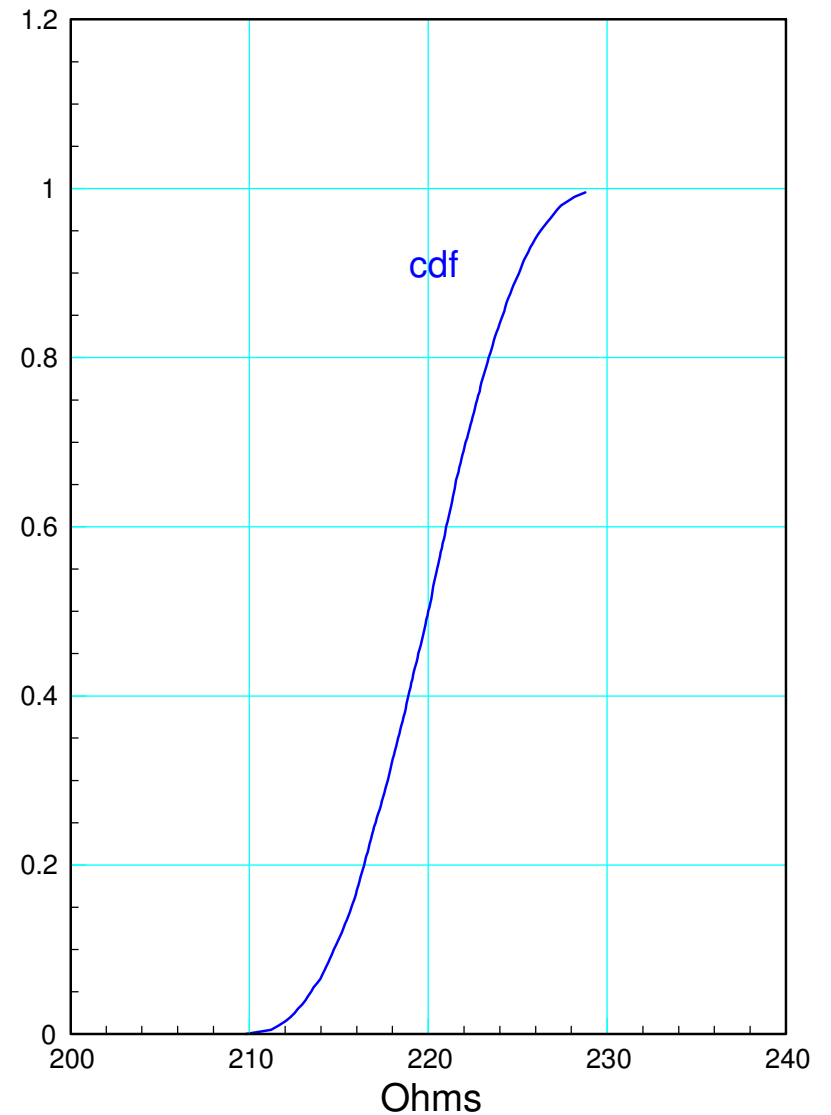
The result is the cdf

```
Data = [];  
  
for i=1:1e4  
    R1 = 100 * (0.95 + 0.1*rand);  
    R2 = 200 * (0.95 + 0.1*rand);  
    R3 = 300 * (0.95 + 0.1*rand);  
    R23 = 1 / (1/R2 + 1/R3);  
    Rab = R1 + R23;  
    Data = [Data; Rab];  
end  
  
Data = sort(Data)  
p = [1:length(Data)]' / length(Data);  
plot(Data, p)  
xlabel('Ohms')
```

Monte-Carlo Result

- cdf
- cdf's all look alike to me...

```
Data = [];  
  
for i=1:1e4  
    R1 = 100 * (0.95 + 0.1*rand);  
    R2 = 200 * (0.95 + 0.1*rand);  
    R3 = 300 * (0.95 + 0.1*rand);  
    R23 = 1 / (1/R2 + 1/R3);  
    Rab = R1 + R23;  
    Data = [Data; Rab];  
end  
  
Data = sort(Data)  
p = [1:length(Data)]' / length(Data);  
plot(Data, p)  
xlabel('Ohms')
```



Monte Carlo for pdf

A little trickier

Create N bins to save the results

- Start from 200 Ohms
- Each bin is 0.1 Ohm wide

Compute 1 million values of Rnet

- Place in each bin

Resulting frequency of bins is pdf

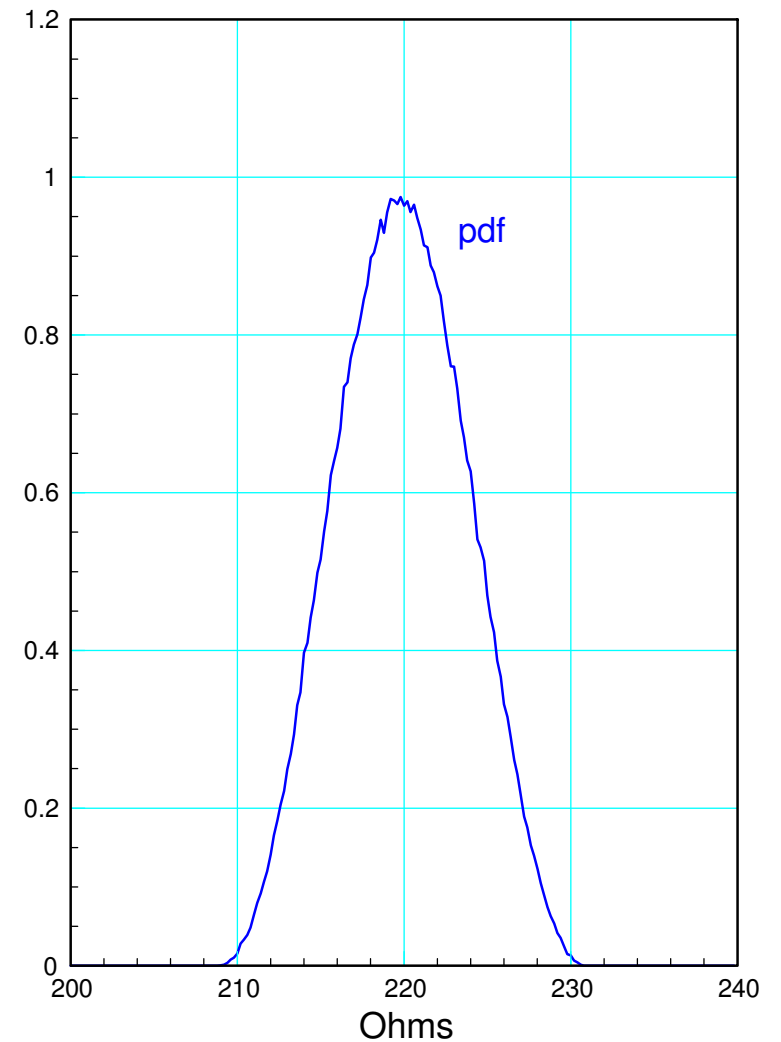
```
X = zeros(2000,1);
Rmin = 200;
for i=1:1e6
    R1 = 100 * (0.95 + 0.1*rand);
    R2 = 200 * (0.95 + 0.1*rand);
    R3 = 300 * (0.95 + 0.1*rand);
    Rnet = R1 + 1 / (1/R2 + 1/R3);
    Bin = round((Rnet-Rmin)*10);
    X(Bin) = X(Bin) + 1;
end
X = X / 1e6;

plot([1:2000]/10 + Rmin, X);
xlabel('Ohms')
```

Resulting pdf

Easier to see what the function looks like

```
X = zeros(2000,1);  
Rmin = 200;  
for i=1:1e6  
    R1 = 100 * (0.95 + 0.1*rand);  
    R2 = 200 * (0.95 + 0.1*rand);  
    R3 = 300 * (0.95 + 0.1*rand);  
    Rnet = R1 + 1 / (1/R2 + 1/R3);  
    Bin = round((Rnet-Rmin)*10);  
    X(Bin) = X(Bin) + 1;  
end  
X = X / 1e6;  
  
plot([1:2000]/10 + Rmin, X);  
xlabel('Ohms')
```

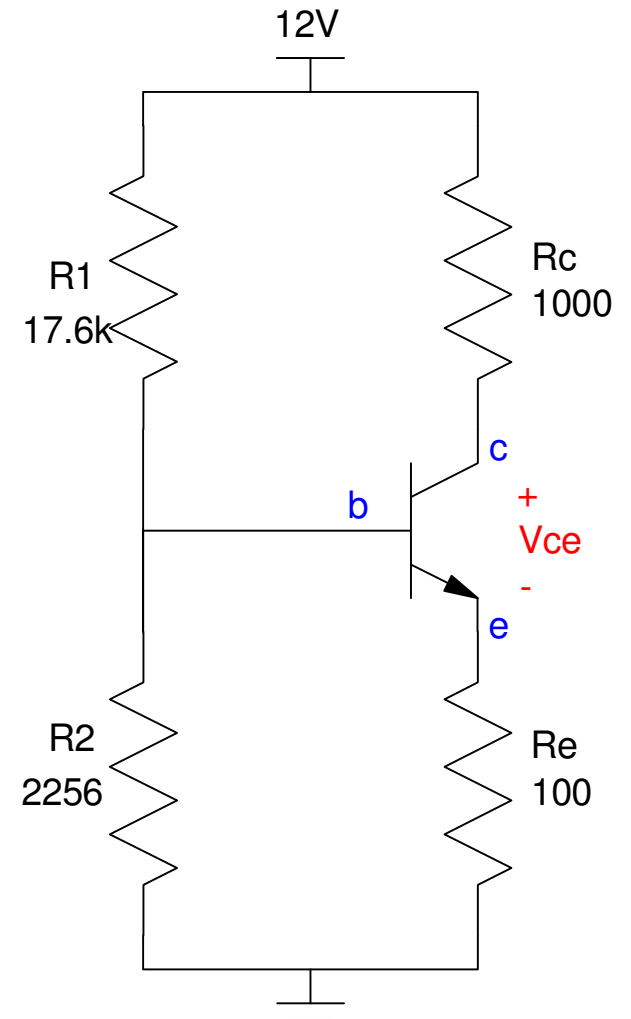


Example 2: BJT Circuit

- Circuit from Electronics

What is V_{ce} assuming

- Each resistor has 5% tolerance
- $100 < \beta < 300$?



Circuit Equations:

- From Electronics

The equations for this circuit

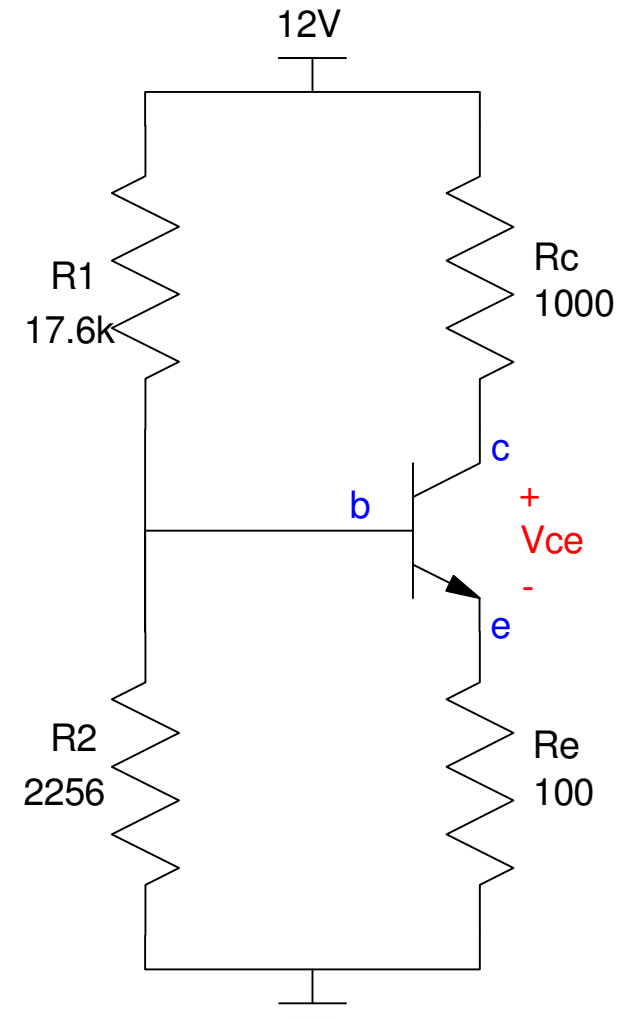
$$V_{th} = \left(\frac{R_2}{R_2 + R_1} \right) 12V$$

$$R_{th} = \left(\frac{R_1 R_2}{R_1 + R_2} \right)$$

$$I_b = \left(\frac{V_{th} - 0.7}{R_{th} + (1 + \beta)R_e} \right)$$

$$I_c = \beta I_b$$

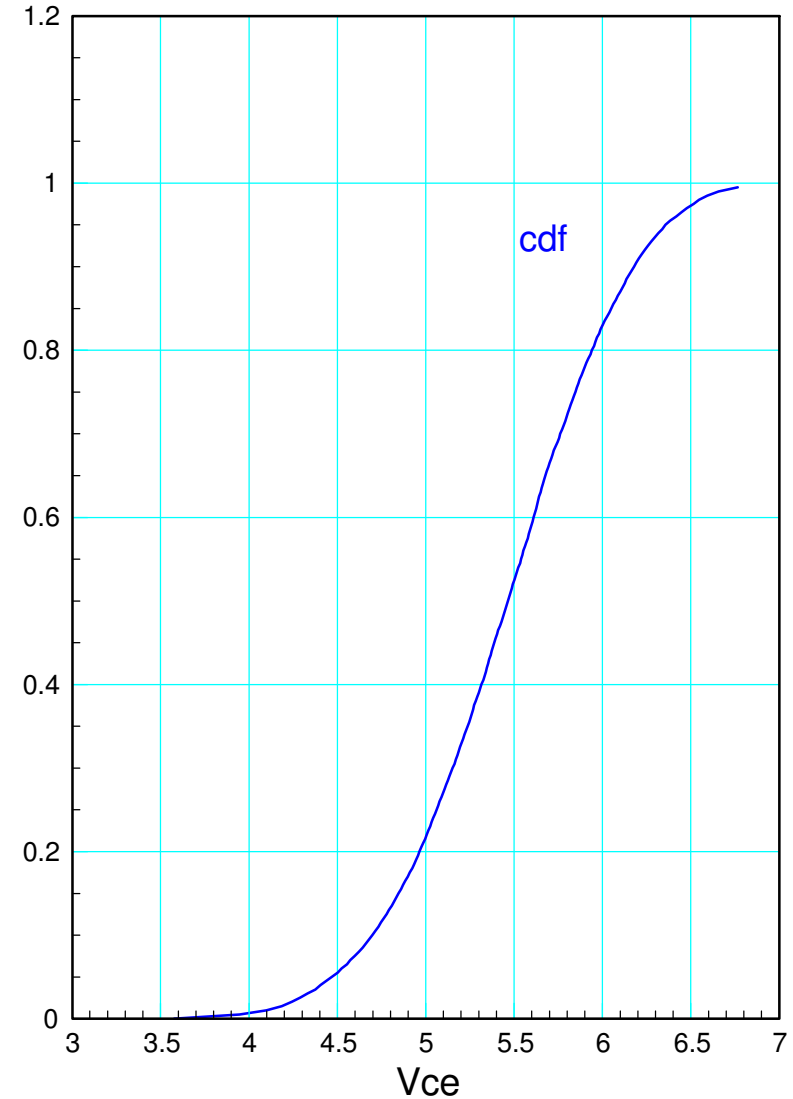
$$V_{ce} = 12 - R_c I_c - R_e (I_b + I_c)$$



cdf from Monte-Carlo

- cdf's all look alike to me...

```
DATA = [];  
  
for i=1:1e4  
    R1 = 17600 * (0.95 + 0.1*rand);  
    R2 = 2256 * (0.95 + 0.1*rand);  
    Rc = 1000 * (0.95 + 0.1*rand);  
    Re = 100 * (0.95 + 0.1*rand);  
    Beta = 100 + 200*rand;  
    Vb = 12*(R2 / (R1+R2));  
    Rb = 1/(1/R1 + 1/R2);  
    Ib = (Vb-0.7) / (Rb + (1+Beta)*Re);  
    Ic = Beta*Ib;  
    Vce = 12 - Rc*Ic - Re*(Ic+Ib);  
    DATA = [DATA; Vce];  
end  
  
DATA = sort(DATA);  
p = [1:length(DATA)]' / length(DATA);  
plot(DATA, p)  
xlabel('Vce')
```



pdf from Monte Carlo

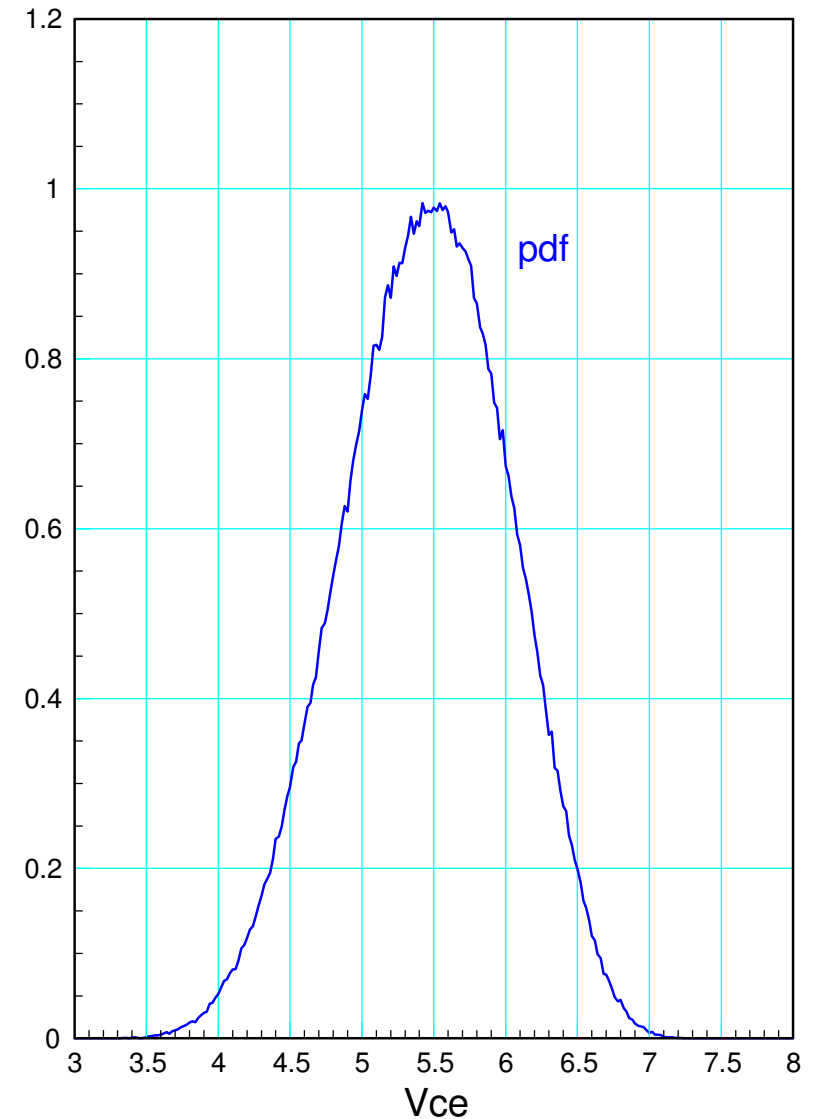
- pdf is more informative

```
Data = zeros(1000,1);

for i=1:1e6
    R1 = 17600 * (0.95 + 0.1*rand);
    R2 = 2256 * (0.95 + 0.1*rand);
    Rc = 1000 * (0.95 + 0.1*rand);
    Re = 100 * (0.95 + 0.1*rand);
    Beta = 100 + 200*rand;
    Vb = 12*(R2 / (R1+R2));
    Rb = 1/(1/R1 + 1/R2);
    Ib = (Vb-0.7) / (Rb + (1+Beta)*Re);
    Ic = Beta*Ib;
    Vce = 12 - Rc*Ic - Re*(Ic+Ib);
    Bin = round(Vce*100);
    Data(Bin) = Data(Bin) + 1;
end

Data = Data / max(Data);

V = [1:1000]'/100;
plot(V, Data);
xlabel('Volts');
```



Summary

Uniform distributions are fairly common and easy to model

- `rand()` function in Matlab

The sum of uniform distributions converges to a Normal distribution

- Central Limit Theorem (coming soon)

Convolution is required to sum uniform distributions

- `conv()` in Matlab
- Or you can use multiplication if using Laplace transforms
 - a.k.a. moment generating functions in statistics

Monte-Carlo is useful when combining uniform distributions other ways

- Circuit analysis
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