
z-Transforms

ECE 341: Random Processes

Lecture #7

note: All lecture notes, homework sets, and solutions are posted on www.BisonAcademy.com

Recap: LaPlace Transforms

LaPlace transforms are a tool which

- Help with the analysis of differential equations, (Math 265)
- Help with the analysis of RLC circuits with analog inputs (ECE 311), and
- Help with the analysis of continuous probability density functions (ECE 341)

LaPlace transforms assume all functions are in the form of

$$y(t) = e^{st}$$

This turns differentiation into multiplication by 's'

$$\frac{dy}{dt} = s \cdot e^{st} = sY$$

'sY' can then be interpreted to mean *the derivative of y*

- LaPlace transforms turn differential equations in to algebraic equations in s.
 - The assumption is that algebra is easier than calculus
-

LaPlace Transforms and Differential Equations

LaPlace Transforms

- Converts differential equations into algebraic equations
- Turns convolution into multiplication

Example, solve for $y(t)$:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 10y = 5\frac{dx}{dt} + 10x$$

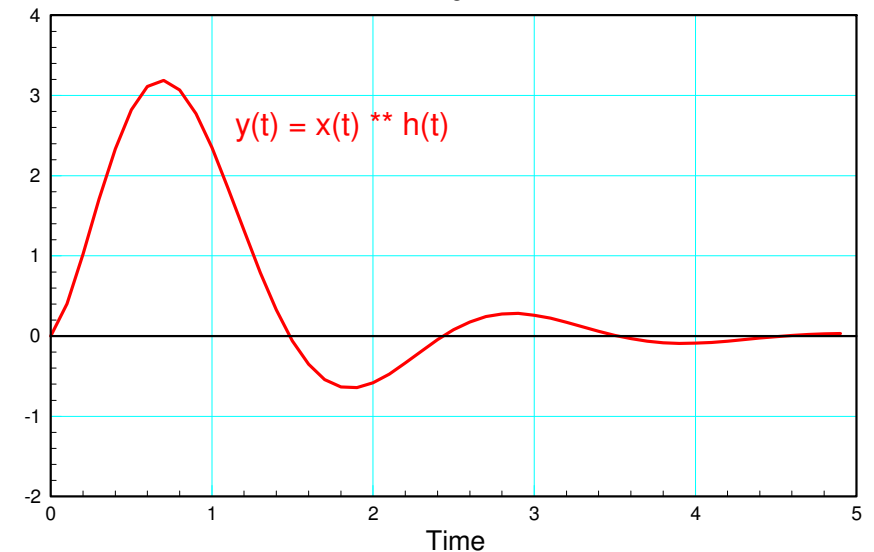
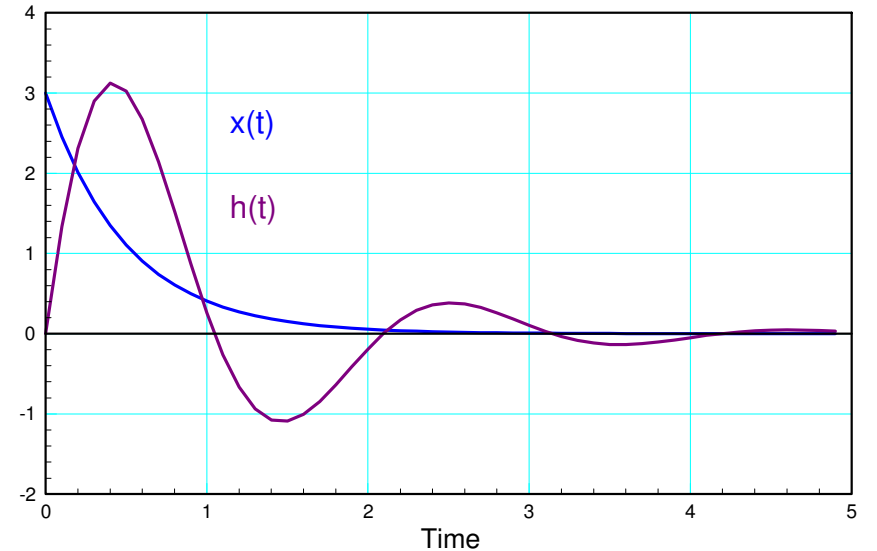
$$x(t) = 3e^{-2t}u(t)$$

Go to the LaPlace domain

$$s^2Y + 2sY + 10Y = 5sX + 10X$$

$$Y = \left(\frac{5s+10}{s^2+2s+10} \right) X = \left(\frac{5s+10}{s^2+2s+10} \right) \left(\frac{3}{s+2} \right)$$

Use a table to find $y(t)$



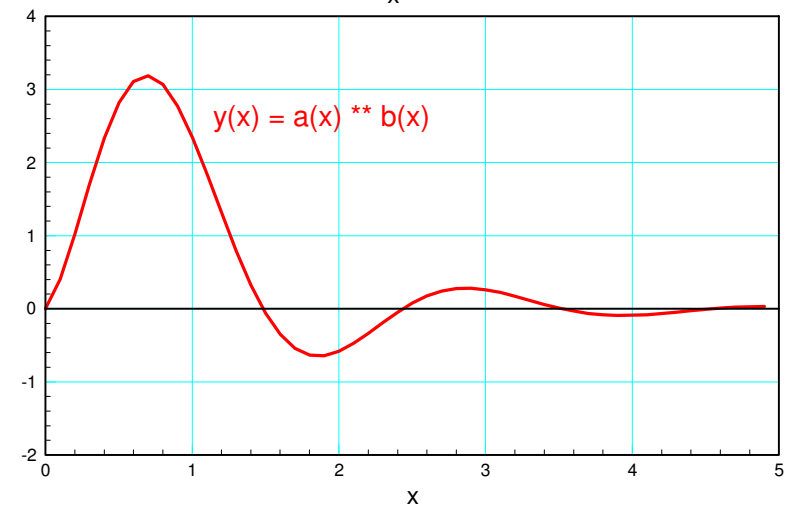
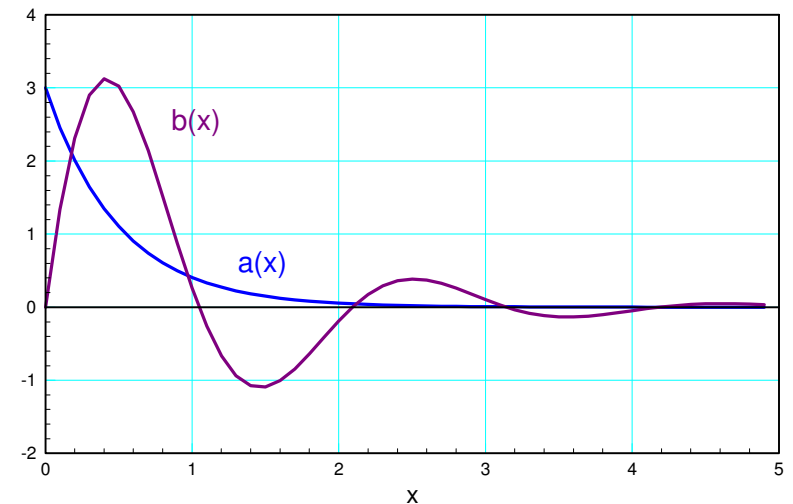
LaPlace Transforms for Random Processes

Similarly,

- If we ever need to convolve to continuous probability density functions (pdf),
- Using the LaPlace transform will convert convolution into multiplication

The LaPlace transform of a pdf is termed its *moment generating function*

- Statistics uses a different name, but it's just the LaPlace transform
- Coming soon in ECE 341 when we get to continuous probability density functions



z-Transforms

z-Transforms are a tool which

- Help with the analysis of difference equations (ECE 434, ECE 376)
- Help with the analysis of RLC circuits with digital inputs (ECE 461), and
- Help with the analysis of discrete probability density functions (ECE 341)

z-transforms assume all functions are in the form of

$$y(k) = z^k$$

This turns a time advance into multiplication by 'z'

$$y(k+1) = z^{k+1} = z \cdot y(k)$$

zY can then be thought of as *the next value of $y(k)$*

z-Transforms and Difference Equations

z-Transforms

- Convert difference equations into algebraic equations in z , and
- Turn discrete-time convolution into multiplication

For example, find $y(k)$

$$y(k+2) - 1.9y(k+1) + 0.9y(k) = 0.02(x(k+1) - x(k))$$

Convert to the z -domain

$$z^2 Y - 1.9zY + 0.9Y = 0.02(zX - X)$$

or

$$Y = \left(\frac{0.02(z-1)}{z^2 - 1.9z + 0.9} \right) X$$

LaPlace and z-Transforms

If you're dealing with continuous functions

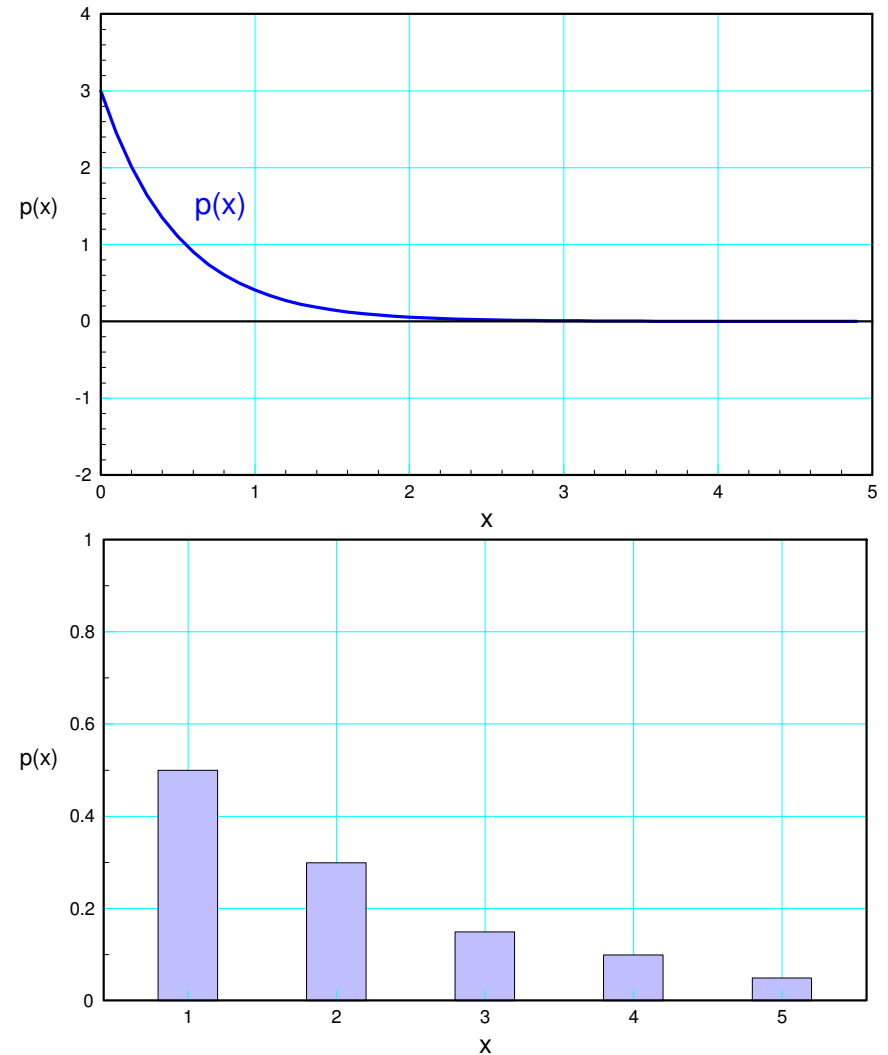
- in time
- in probability

use LaPlace transforms.

If you're dealing with discrete functions

- in time
- in probability

use z-Transforms



z-Transform Properties:

- www.wikipedia.com

The z-transform is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x_n \cdot z^{-n}$$

Linearity:

$$Z(ax_n + by_n) = aX(z) + bY(z)$$

Proof: The z-transform is

$$Z(ax_n + by_n) = \sum_{n=-\infty}^{\infty} (ax_n + by_n) \cdot z^{-n}$$

$$Z(ax_n + by_n) = \left(a \sum_{n=-\infty}^{\infty} x_n \cdot z^{-n} \right) + \left(b \sum_{n=-\infty}^{\infty} y_n \cdot z^{-n} \right)$$

$$Z(ax_n + by_n) = aX(z) + bY(z)$$

Time Shifting:

$$Z(x_{n-k}) = z^{-k} \cdot X(z)$$

Proof:

$$Z(x_{n-k}) = \sum_{n=-\infty}^{\infty} x_{n-k} \cdot z^{-n}$$

Let $m = n-k$

$$Z(x_{n-k}) = \sum_{m=-\infty}^{\infty} x_m \cdot z^{-(m+k)}$$

$$Z(x_{n-k}) = \sum_{m=-\infty}^{\infty} x_m \cdot z^{-m} \cdot z^{-k}$$

$$Z(x_{n-k}) = z^{-k} \cdot \left(\sum_{m=-\infty}^{\infty} x_m \cdot z^{-m} \right)$$

$$Z(x_{n-k}) = z^{-k} \cdot X(z)$$

Multiplying by $1/z$ means delay the signal by one.

Convolution:

$$Z(x_n * y_n) = X(z) \cdot Y(z)$$

Proof:

$$Z\left(\sum_{k=-\infty}^{\infty} x_k \cdot y_{n-k}\right) = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x_k \cdot y_{n-k}\right) \cdot z^{-n}$$

Change the order of summation:

$$= \sum_{k=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} x_k \cdot y_{n-k}\right) \cdot z^{-n} = \left(\sum_{k=-\infty}^{\infty} x_k \left(\sum_{n=-\infty}^{\infty} y_{n-k}\right)\right) \cdot z^{-n}$$

Let $m = n-k$

$$\begin{aligned} &= \left(\sum_{k=-\infty}^{\infty} x_k \left(\sum_{n=-\infty}^{\infty} y_m\right)\right) \cdot z^{-(m+k)} = \left(\left(\sum_{k=-\infty}^{\infty} x_k \cdot z^{-k}\right) \left(\sum_{n=-\infty}^{\infty} y_m \cdot z^{-m}\right)\right) \\ &= X(z) \cdot Y(z) \end{aligned}$$

This is a biggie - z-transforms turn convolution into multiplication.

Table of z-Transforms:

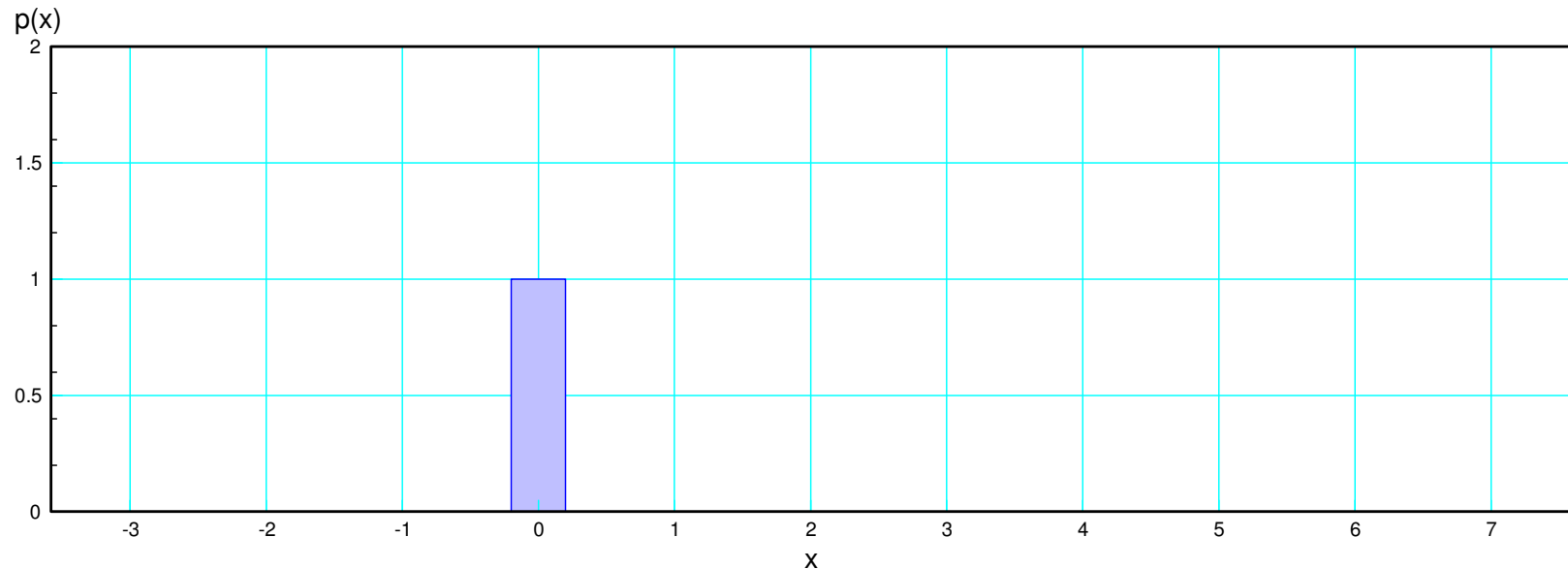
function	$y(k) \ (k > 0)$	$Y(z)$
delta	$\delta(k) = \begin{cases} 1 & k = 0 \\ 0 & \text{otherwise} \end{cases}$	1
unit step	$u(k) = 1$	$\left(\frac{z}{z-1} \right)$
ramp	k	$\left(\frac{z}{(z-1)^2} \right)$
parabola	k^2	$\left(\frac{z(z+1)}{(z-1)^3} \right)$
cubic	k^3	$\left(\frac{z(z^2+4z+1)}{(z-1)^4} \right)$
decaying exponential	a^k	$\left(\frac{z}{z-a} \right)$
	$k a^k$	$\left(\frac{za}{(z-a)^2} \right)$
	$k^2 a^k$	$\left(\frac{az(z+a)}{(z-a)^3} \right)$
damped sinewave	$2b \cdot a^k \cdot \cos(k\theta + \phi) \cdot u(k)$	$\left(\frac{(b\angle\phi)z}{z-(a\angle\theta)} \right) + \left(\frac{(b\angle-\phi)z}{z-(a\angle-\theta)} \right)$

Proof: Delta Function. This is sort-of the definition of z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x_n \cdot z^{-n} = \dots + x_0 \cdot z^0 + x_1 \cdot z^1 + x_2 \cdot z^2 + \dots$$

If $x(k)$ is a delta function:

$$X(z) = 1$$

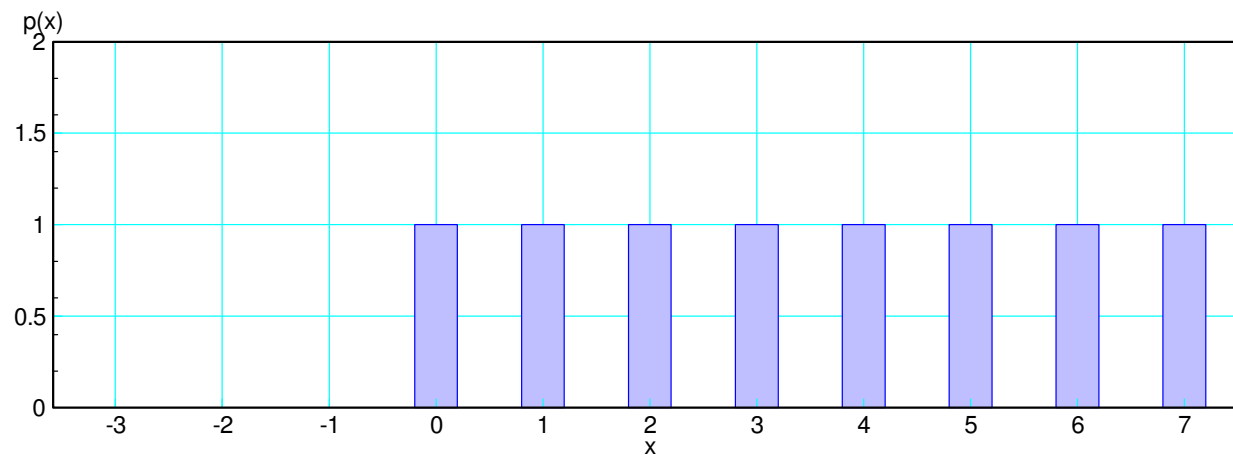


Proof: Unit Step. Using a table:

	z^2	z^1	z^0	z^{-1}	z^{-2}	z^{-3}	z^{-4}
$X(z)$	0	0	1	1	1	1	1
$z^{-1} X(z)$	0	0	0	1	1	1	1
subtract							
$\left(1 - \frac{1}{z}\right) X(z)$	0	0	1	0	0	0	0

so

$$X(z) = \frac{1}{\left(1 - \frac{1}{z}\right)} = \left(\frac{z}{z-1}\right)$$

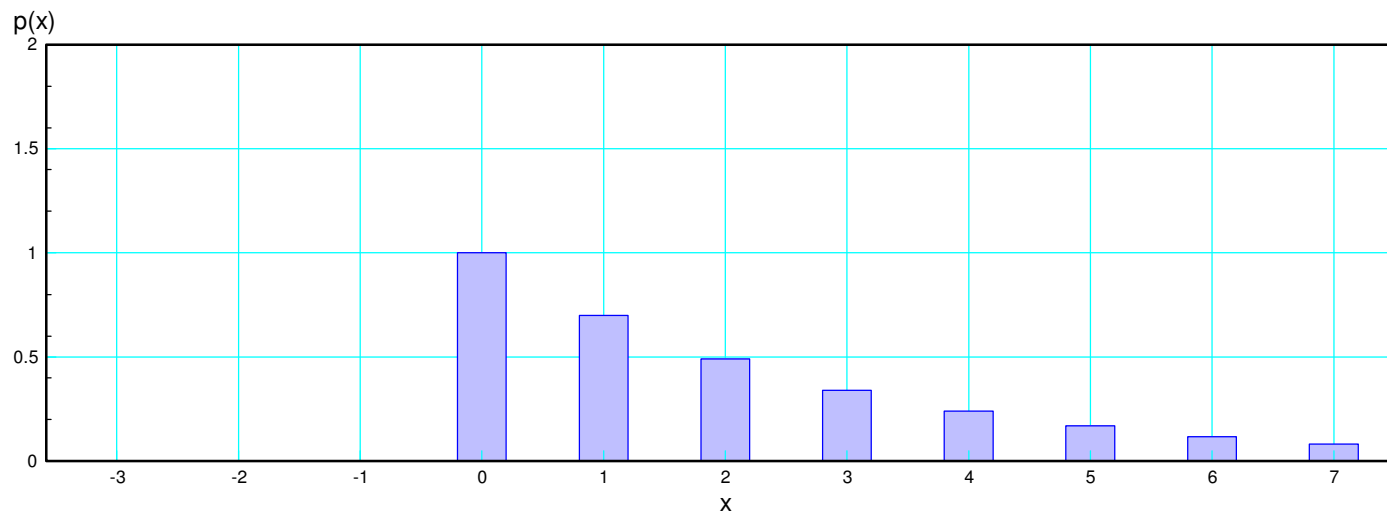


Proof: Decaying Exponential. Using a table:

	z^2	z^1	z^0	z^{-1}	z^{-2}	z^{-3}	z^{-4}
$X(z)$	0	0	1	a	a^2	a^3	a^4
$a * z^{-1} X(z)$	0	0	0	a	a^2	a^3	a^4
subtract							
$(1 - \frac{a}{z})X(z)$	0	0	1	0	0	0	0

SO

$$X(z) = \left(\frac{1}{1 - \frac{a}{z}} \right) = \left(\frac{z}{z - a} \right)$$



Solving Functions in the z-Domain

Problem 1: Find the step response of

$$Y = \left(\frac{0.2z}{(z-0.9)(z-0.5)} \right) X$$

i) Replace $X(z)$ with the z-transform of a step

$$Y = \left(\frac{0.2z}{(z-0.9)(z-0.5)} \right) \left(\frac{z}{z-1} \right)$$

ii) Use partial fractions (pull out a z - we'll need this)

$$Y = \left(\frac{0.2z}{(z-1)(z-0.9)(z-0.5)} \right) z$$

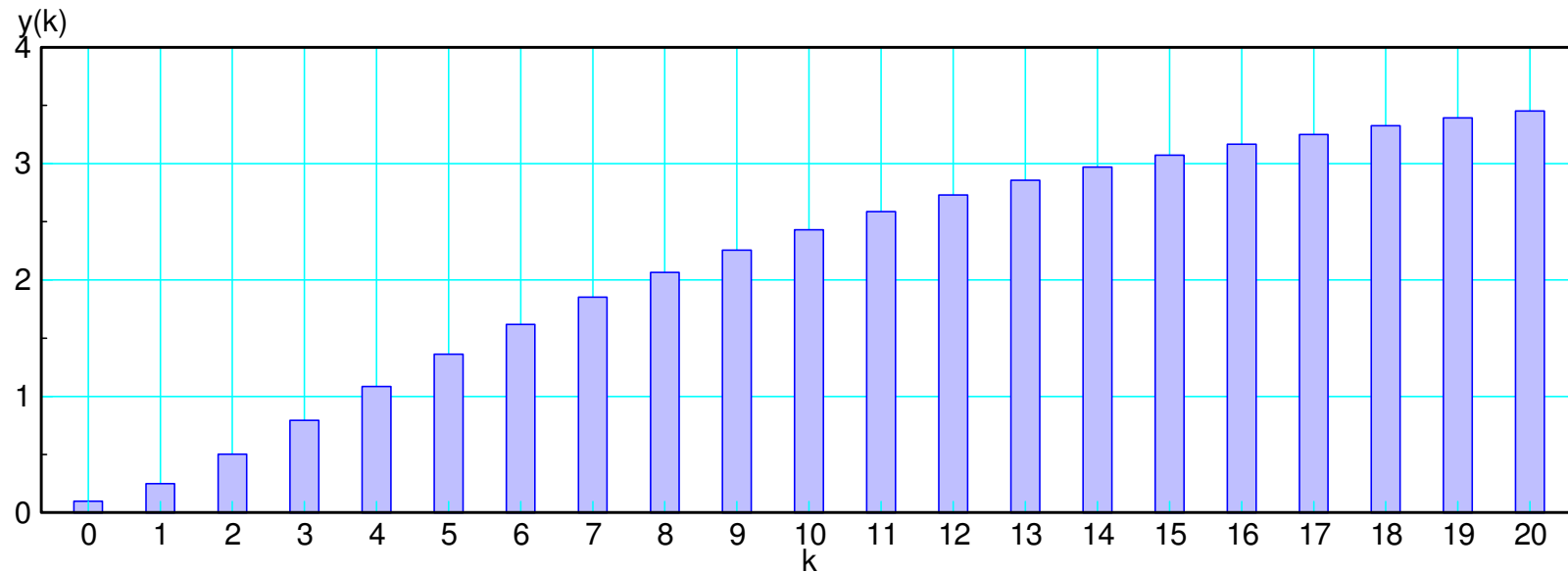
$$Y = \left(\left(\frac{4}{z-1} \right) + \left(\frac{-4.5}{z-0.9} \right) + \left(\frac{0.5}{z-0.5} \right) \right) z$$

Multiply through by z

$$Y = \left(\left(\frac{4z}{z-1} \right) + \left(\frac{-4.5z}{z-0.9} \right) + \left(\frac{0.5z}{z-0.5} \right) \right)$$

iii) Now apply the table entries

$$y(k) = 4 - 4.5 \cdot (0.9)^k + 0.5 \cdot (0.5)^k \quad k \geq 0$$



Problem 2: Find the step response of a system with complex poles:

$$Y = \left(\frac{0.2z}{(z-0.9\angle 10^0)(z-0.9\angle -10^0)} \right) X$$

i) Replace X with its z-transform (a unit step)

$$Y = \left(\frac{0.2z}{(z-0.9\angle 10^0)(z-0.9\angle -10^0)} \right) \left(\frac{z}{z-1} \right)$$

ii) Factor out a z and use partial fractions

$$Y = \left(\left(\frac{5.355}{z-1} \right) + \left(\frac{2.98\angle 153.97^0}{z-0.9\angle 10^0} \right) + \left(\frac{2.98\angle -153.97^0}{z-0.9\angle -10^0} \right) \right) z$$

iii) Convert back to time using the table of z-transforms

$$y(k) = 5.355 + 4.859 \cdot (0.9)^k \cdot \cos(10^0 \cdot k - 153.97^0) \quad k \geq 0$$

Trick if the numerator does not have a z-term

Find the inverse z-transform for

$$Y = \left(\frac{0.2}{(z-1)(z-0.9)(z-0.5)} \right)$$

Multiply by z:

$$zY = \left(\frac{0.2}{(z-1)(z-0.9)(z-0.5)} \right) z$$

Do partial fractions

$$zY = \left(\frac{4}{z-1} - \frac{5}{z-0.9} + \frac{1}{z-0.5} \right) z$$

$$zY = \left(\frac{4z}{z-1} - \frac{5z}{z-0.9} + \frac{z}{z-0.5} \right)$$

Take the inverse z-transform

$$zY = \left(\frac{4z}{z-1} - \frac{5z}{z-0.9} + \frac{z}{z-0.5} \right)$$

$$zy(k) = \left(4 - 5(0.9)^k + (0.5)^k \right) u(k)$$

Divide by z

- time shift: delay by one

$$y(k) = \left(4 - 5(0.9)^{k-1} + (0.5)^{k-1} \right) u(k-1)$$

or equivalently

$$y(k) = \left(4 - 5.555(0.9)^k + 2(0.5)^k \right) u(k-1)$$

Problem 3: Repeated Poles

Find the inverse z-transform for

$$X(z) = \left(\frac{0.2(z+1)}{(z-1)(z-0.9)^2} \right)$$

Repeated Poles

Option 1: With repeated poles, there will be three terms

$$\left(\frac{0.2(z+1)}{(z-1)(z-0.9)^2} \right) = \left(\frac{a}{z-1} \right) + \left(\frac{b}{(z-0.9)^2} \right) + \left(\frac{c}{z-0.9} \right)$$

a and b can be found using the cover-up method

$$a = \left(\frac{0.2(z+1)}{(z-0.9)^2} \right)_{z=1} = 40$$

$$b = \left(\frac{0.2(z+1)}{(z-1)} \right)_{z=0.9} = -3.8$$

Repeated Poles

c can be found by placing over a common denominator

$$\left(\frac{0.2(z+1)}{(z-1)(z-0.9)^2} \right) = \left(\frac{a}{z-1} \right) + \left(\frac{b}{(z-0.9)^2} \right) + \left(\frac{c}{z-0.9} \right)$$

$$\left(\frac{0.2(z+1)}{(z-1)(z-0.9)^2} \right) = \left(\frac{a}{z-1} \right) \left(\frac{(z-0.9)^2}{(z-0.9)^2} \right) + \left(\frac{b}{(z-0.9)^2} \right) \left(\frac{z-1}{z-1} \right) + \left(\frac{c}{z-0.9} \right) \left(\frac{z-1}{z-1} \right) \left(\frac{z-0.9}{z-0.9} \right)$$

Matching the numerator terms

$$0.2(z+1) = a(z-0.9)^2 + b(z-1) + c(z-1)(z-0.9)$$

The z^2 term has to match

$$0 = az^2 + cz^2$$

$$c = -a$$

$$\left(\frac{0.2(z+1)}{(z-1)(z-0.9)^2} \right) = \left(\frac{40}{z-1} \right) + \left(\frac{-3.8}{(z-0.9)^2} \right) + \left(\frac{-40}{z-0.9} \right)$$

Now take the inverse-z transform

$$X = \left(\frac{40}{z-1} \right) + \left(\frac{-3.8}{(z-0.9)^2} \right) + \left(\frac{-40}{z-0.9} \right)$$

$$zX = \left(\frac{40z}{z-1} \right) + \left(\frac{-3.8z}{(z-0.9)^2} \right) + \left(\frac{-40z}{z-0.9} \right)$$

$$zx(k) = \left(40 - 3.8k(0.9)^k - 40(0.9)^k \right) u(k)$$

Divide by z

$$x(k) = \left(40 - 3.8(k-1)(0.9)^{k-1} - 40(0.9)^{k-1} \right) u(k-1)$$

Repeated Poles (take 2)

You can also solve by first multiplying both sides by z^2 to make the numerator and denominator both 3rd order

$$z^2 X(z) = \left(\frac{0.2z^2(z+1)}{(z-1)(z-0.9)^2} \right)$$

Pull out a z

$$z^2 X = \left(\frac{0.2z(z+1)}{(z-1)(z-0.9)^2} \right) z$$

Do a partial fraction expansion

$$z^2 X = \left(\left(\frac{a}{z-1} \right) + \left(\frac{b}{(z-0.9)^2} \right) + \left(\frac{c}{z-0.9} \right) \right) z$$

time passes....

$$z^2 X = \left(\left(\frac{40}{z-1} \right) + \left(\frac{-3.42}{(z-0.9)^2} \right) + \left(\frac{-39}{z-0.9} \right) \right) z$$

Repeated Poles (cont'd)

$$z^2 X = \left(\frac{40z}{z-1} \right) + \left(\frac{-3.42z}{(z-0.9)^2} \right) + \left(\frac{-39z}{z-0.9} \right)$$

Take the inverse z-transform

$$z^2 x(k) = \left(40 - 3.42k(0.9)^k - 39(0.9)^k \right) u(k)$$

Divide by z^2

$$x(k) = \left(40 - 3.42(k-2)(0.9)^{k-2} - 39(0.9)^{k-2} \right) u(k-2)$$

Both answers are the same

Time Value of Money

Borrow \$100,000 at 6% interest for 10 years.

- What are the monthly payments?

Solution:

- $x(k)$ is how much money you owe
- $x(k+1)$ is how much you owe next month (p = monthly payment):

$$x(k+1) = 1.005x(k) - p + X(0) \cdot \delta(k)$$

Take the z-transform (payments start at month #1 rather #0)

$$zX = 1.005X - p\left(\frac{1}{z-1}\right) + X(0)$$

Solve

$$X = \left(\frac{X(0)}{z-1.005} \right) - p \left(\frac{1}{(z-1)(z-1.005)} \right)$$

Using partial fractions

$$X = \left(\frac{X(0)}{z-1.005} \right) + p \left(\left(\frac{200}{z-1} \right) - \left(\frac{200}{z-1.005} \right) \right)$$

$$zX = \left(\frac{X(0)z}{z-1.005} \right) + p \left(\left(\frac{200z}{z-1} \right) - \left(\frac{200z}{z-1.005} \right) \right)$$

Converting back to the time domain

$$zx(k) = 1.005^k X(0) - 200p(1.005^k - 1)u(k)$$

$$x(k) = 1.005^{k-1} X(0) - 200p(1.005^{k-1} - 1)u(k-1)$$

For 120 payments (10 years)

- The balance at month #121 is \$0
- (no more payments)

$$x(121) = 0 = 1.005^{120}X(0) - 200p(1.005^{120} - 1)$$

$$x(121) = 0 = \$181,939 - 163.879p$$

$$p = \$1110.2050$$

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Loan Calculator

Amortized Loan: Paying Back a Fixed Amount Periodically

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Loan Amount

\$100,000

Loan Term

10 years0 months

Interest Rate

6%

Compound

Monthly (APR) ▾

Pay Back

Every Month ▾

Calculate

Clear

Results: 

Payment Every Month

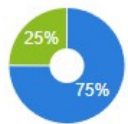
\$1,110.21

Total of 120 Payments

\$133,224.60

Total Interest

\$33,224.60



Principal

Interest

[View Amortization Table](#)

Summary

z-Transforms are similar to LaPlace transforms, but they deal with

- Discrete-time systems
- Discrete probability functions.

When dealing with difference equations or discrete-time events, z-transforms will be useful.

In Signals and Systems, $X(z)$ represents a signal

- Its value at $z = 1$ can be anything

In Random Processes, $X(z)$ represents a probability density function

- $0 \leq x(k) \leq 1$ (probabilities can't be negative and must sum to one)
-