Natural Response of RL Circuits

ECE 211 Circuits I

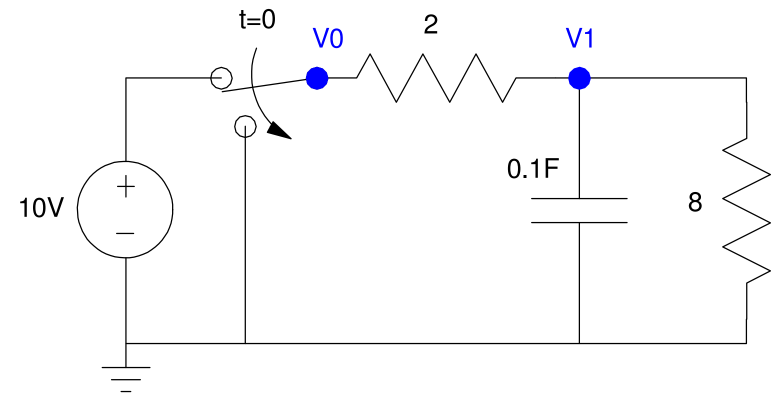
Lecture #36

Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

Last Lecture

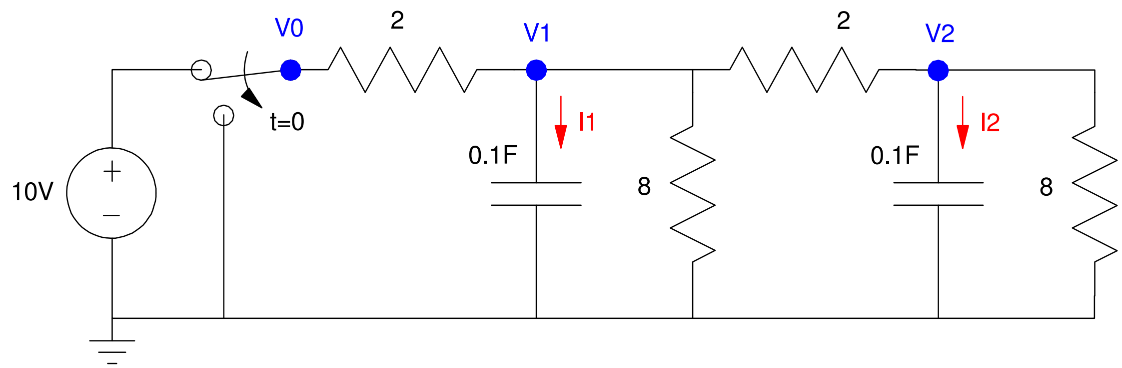
Looked at finding the natural response for

- 1-stage RC circuits and
- 2-stage RC circuits



The procedure was to

- Find the intial conditions
- Find the differential equations
  - Use the p-operator
  - Impedance of C is 1/Cp



Then solve

- Using CircuitLab
- Using calculus
- Using eigenvalues & eigenvectors

This Lecture

Find the natural response for

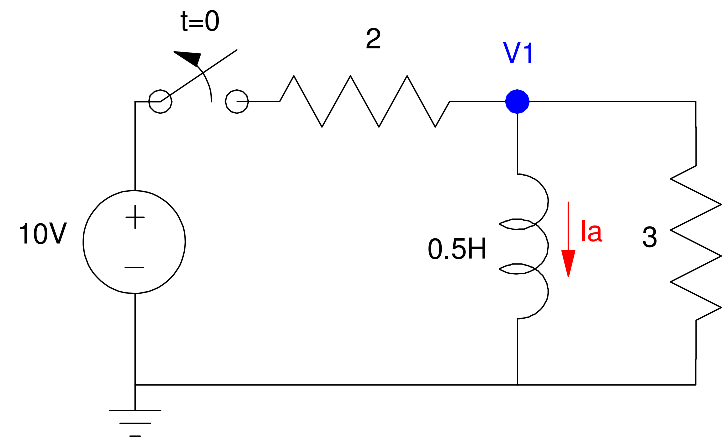
- 1-stage RL circuits and
- 2-stage RL circuits

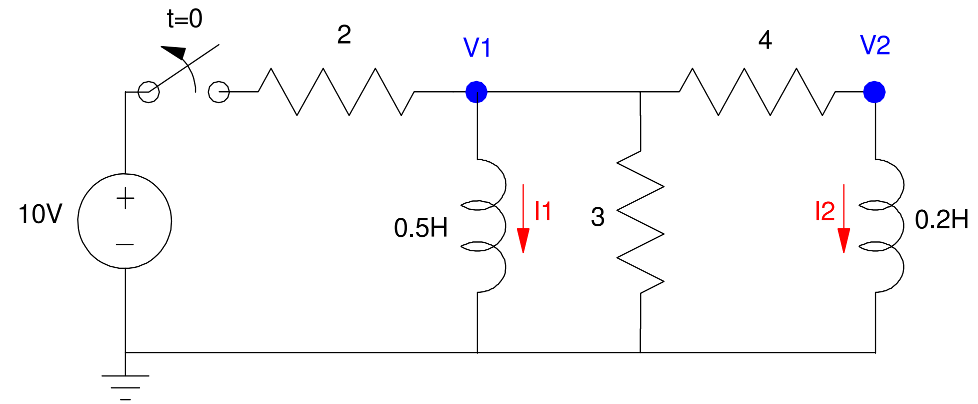
The procedure will be

- Find the intial conditions
- Find the differential equations
    - Use the p-operator
    - Impedance of L is Lp

Then solve

- Using CircuitLab
- Using calculus
- Using eigenvalues & eigenvectors





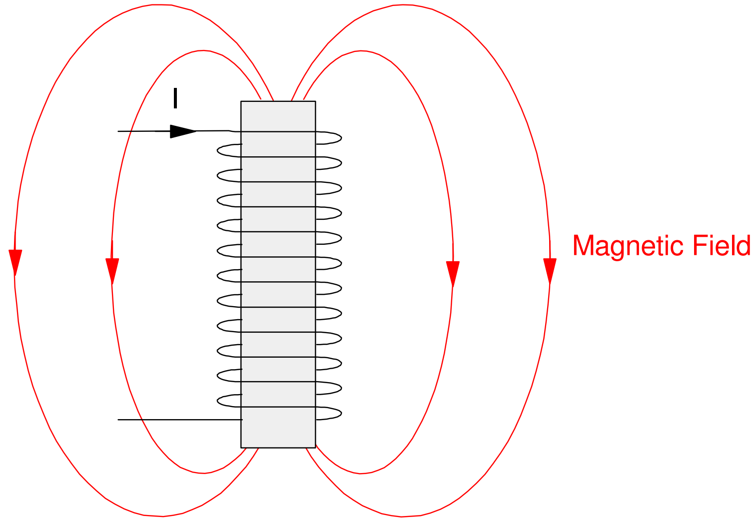
Inductor VI characteristics

For capacitors

$$i = C\frac{dV}{dt}$$

For indictors:

$$v = L\frac{di}{dt}$$



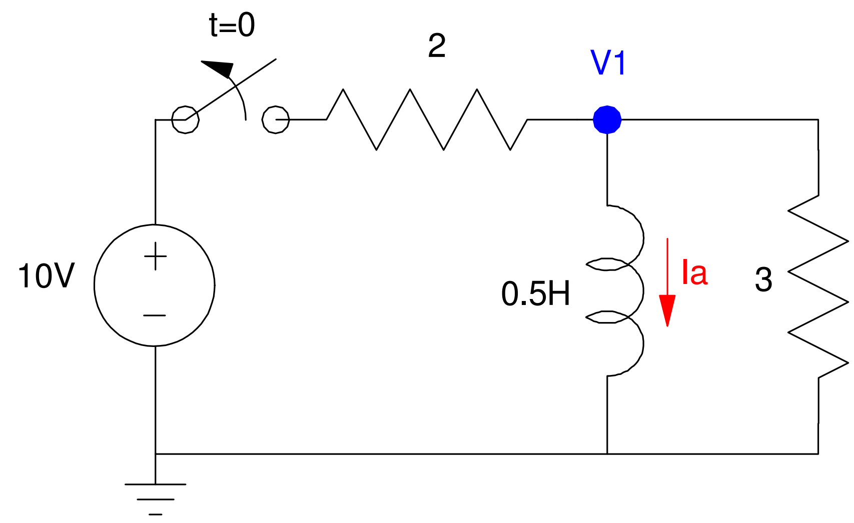
This means that

- Each inductor adds a 1st-order differential equation
- A circuit with N inductors is described by an Nth-order differential equation

1-Stage RL Circuit

For this circuit, find

- The initial condition
  - Ia(0)
- The differential equation
  - Use the p-operator
- The solution to
  - Ia(t)
  - V1(t)

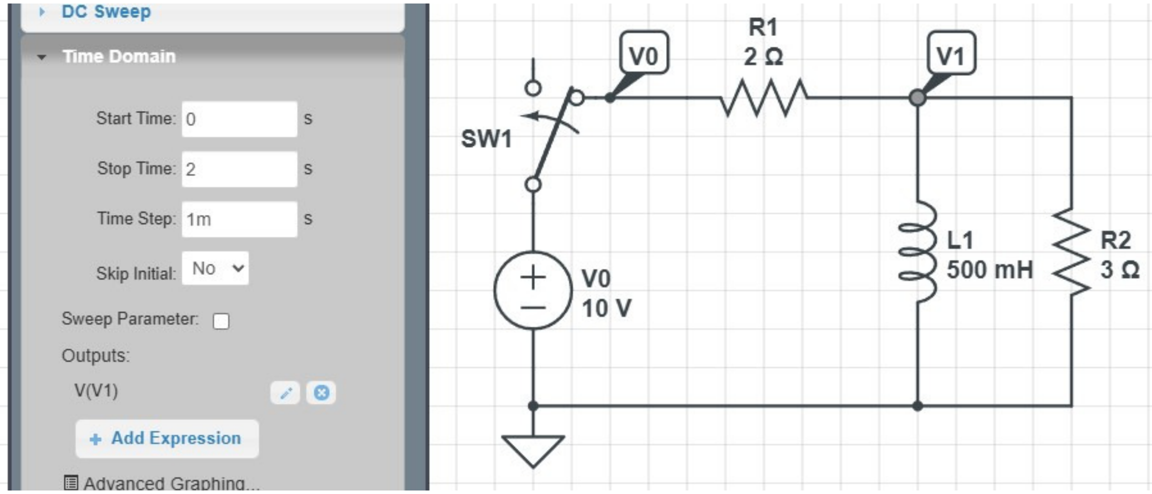


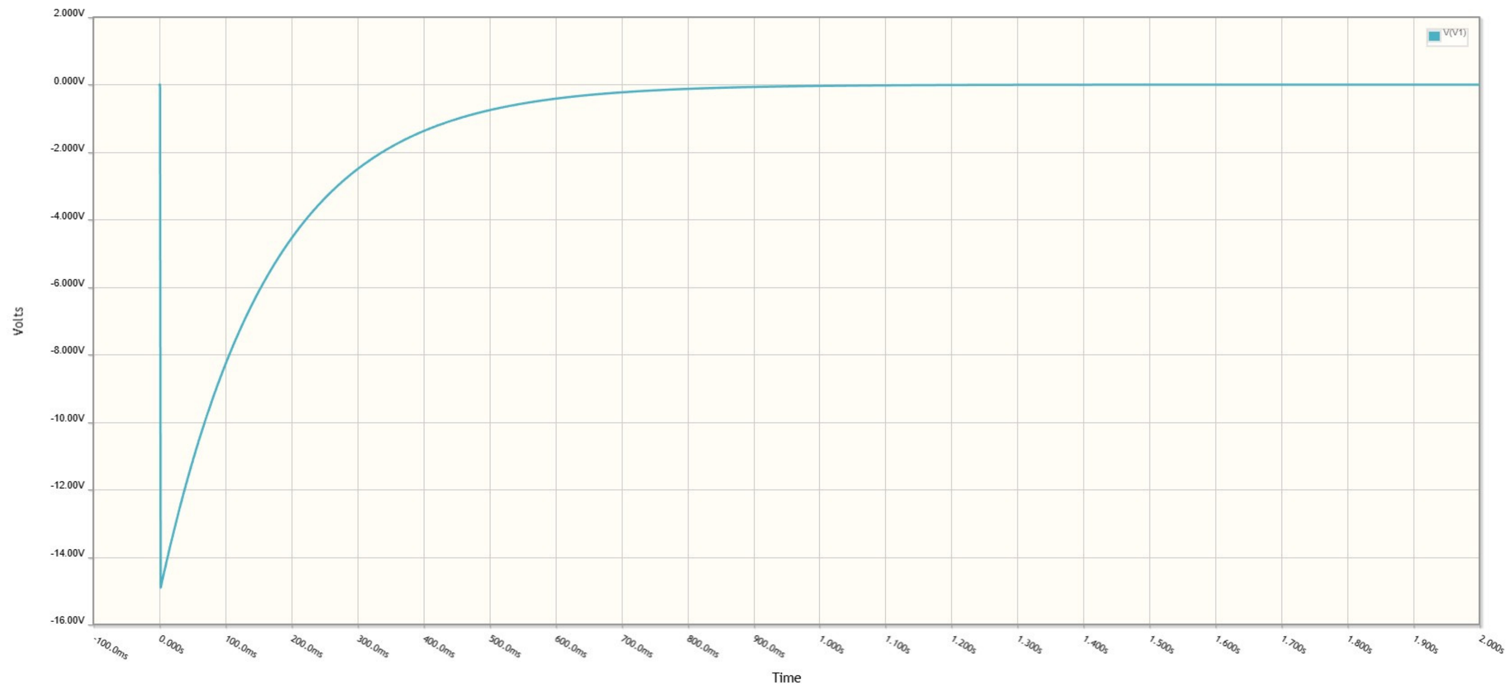
CircuitLab

SW1 opens at t=0

V1(t)

- jumps to -15V
- then decays to 0V





Initial Condition

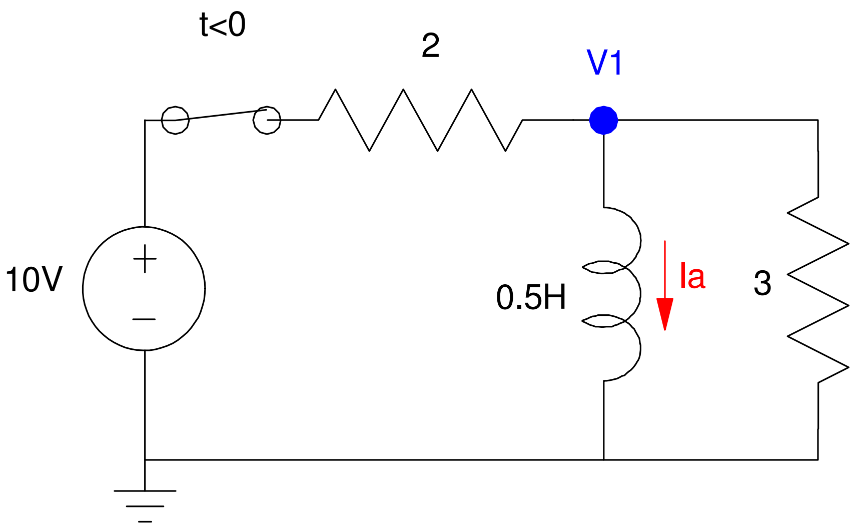
For t<0, the switch is closed

Inductors are shorts at DC

- $V = L\frac{dI}{dt}$
- If Ia(t) is a constant
- V1(t) = 0

Then

$$I_a(0) = \left(\frac{10V - 0V}{2\Omega}\right) = 5A$$

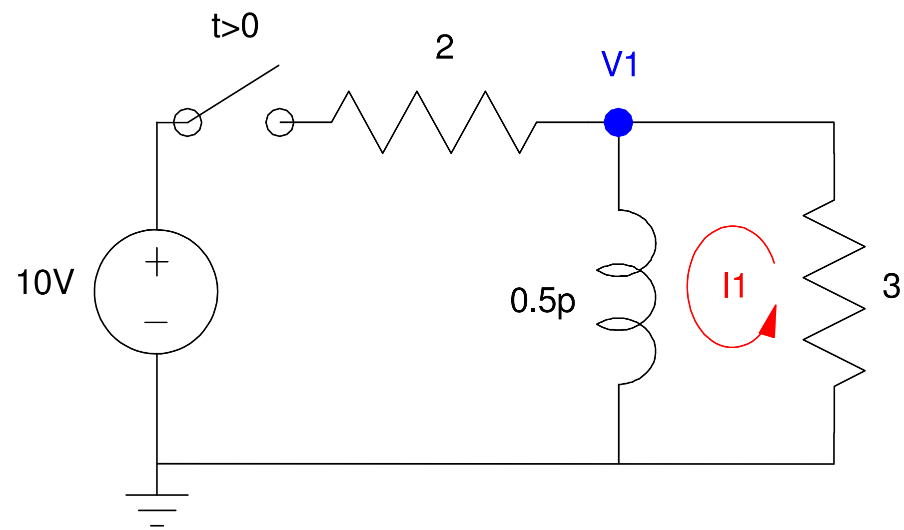


Differential Equation

For t>0, write the current-loop equations

- Replace L with Lp
- Write the current-loop equation

$$(0.5p)I_1 + (3)I_1 = 0$$



Simplify

$$(0.5p + 3)I_1 = 0$$

$$(p + 6)I_1 = 0$$

I1(t)

From

$$(p+6)I_1 = 0$$

$$I_1(0) = 5$$

you get

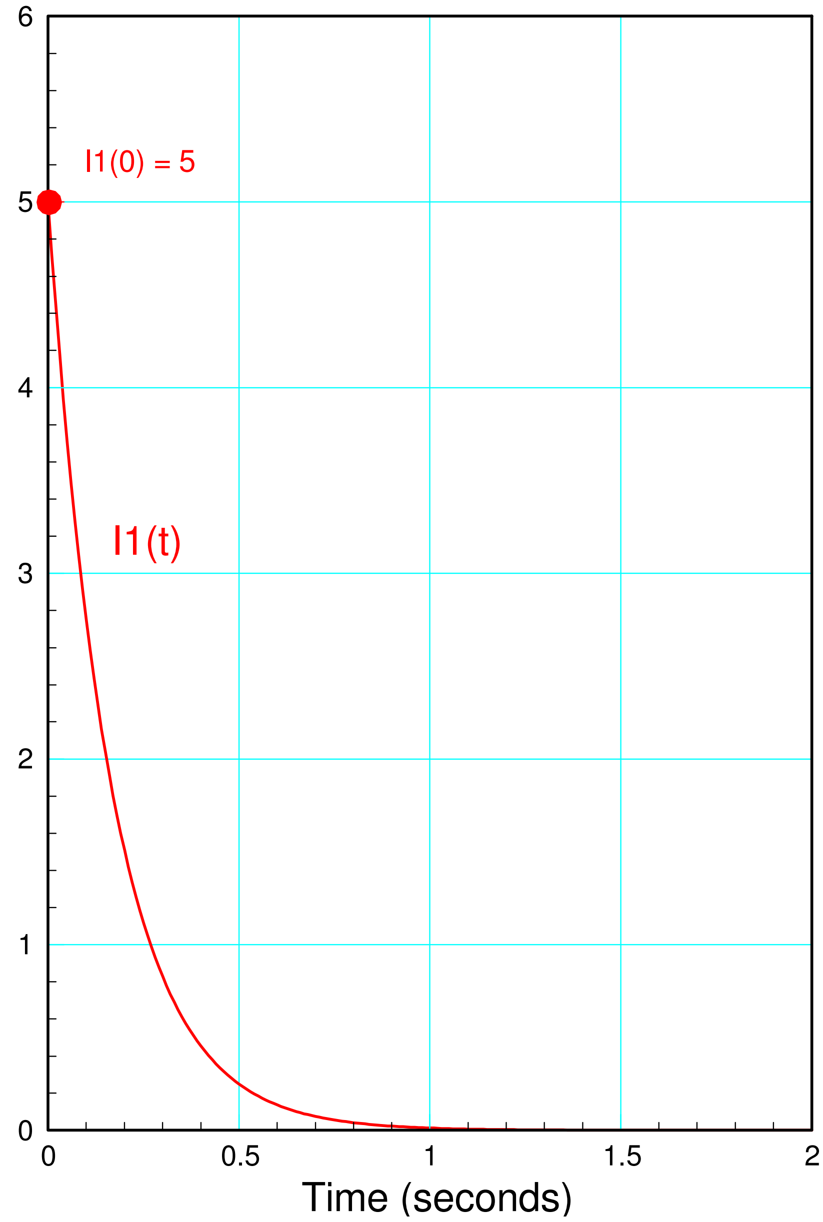
$$p = -6$$

$$I_1(t) = ae^{-6t}$$

$$I_1(0) = 5 = a$$

so

$$I_1(t) = 5e^{-6t}$$



V1(t)

For this circuit

$$V_1 = -3I_1$$

$$V_1(t) = -15e^{-6t}$$

You can also find this as

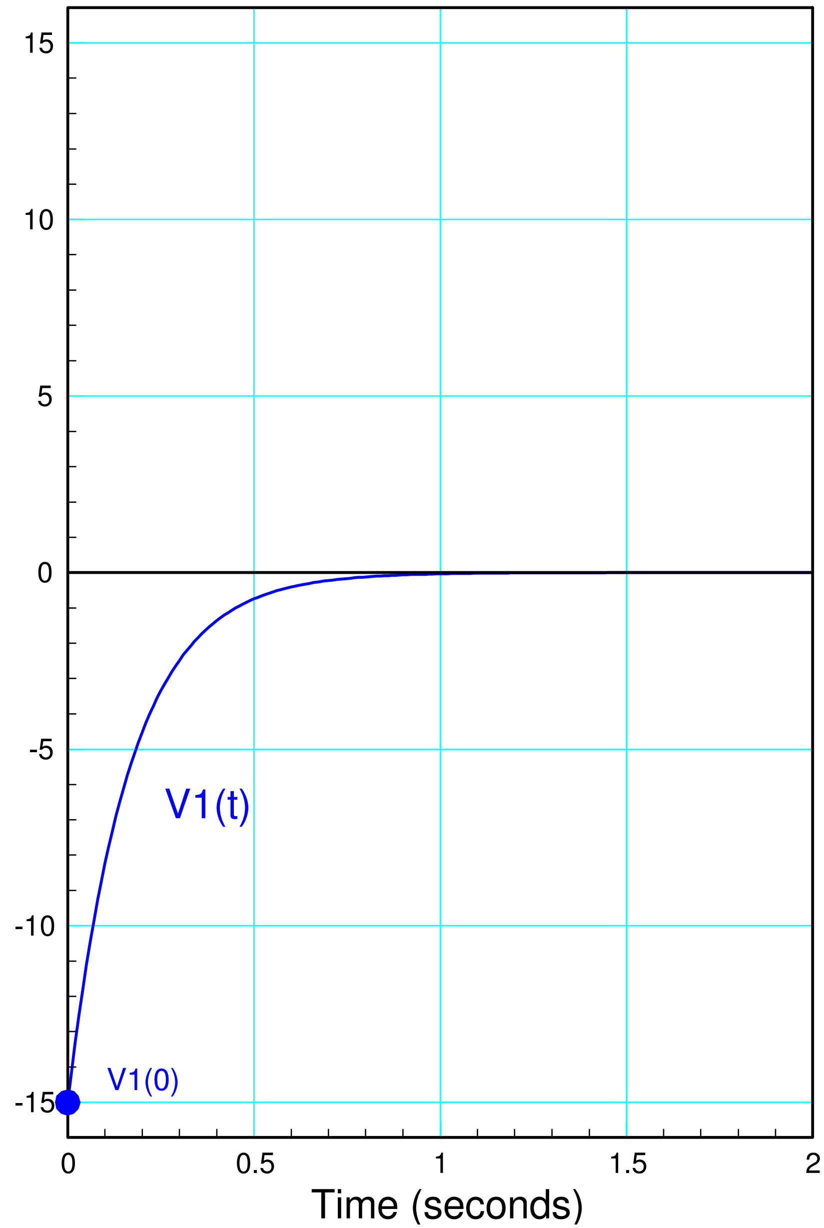
$$V = L\frac{dI}{dt}$$

$$V_1(t) = 0.5 \cdot \frac{d}{dt}(5e^{-6t})$$

$$V_1(t) = -15e^{-6t}$$

Note that 10V in produced -15V out

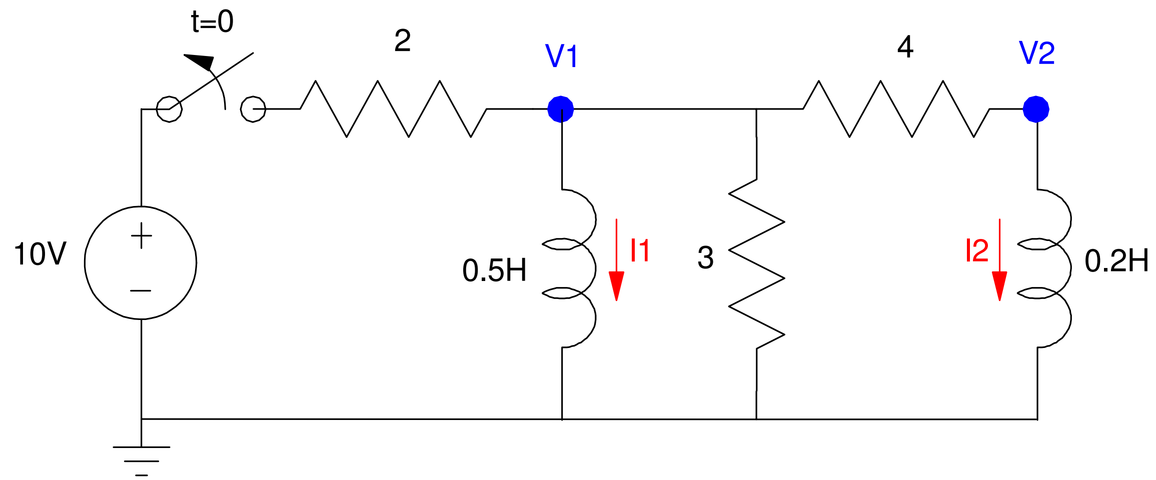
- Voltage inversion
- Voltage amplification
- Useful for changing DC voltages
  - Covered in Electronics



2-Stage RL Circuit

For this circuit, find

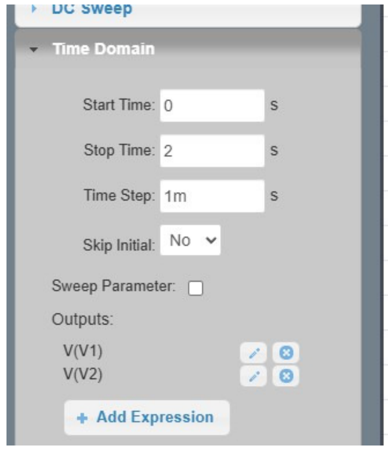
- The initial condition
  - Ia(0), Ib(0)
- The differential equation
  - Use the p-operator
- The solution to
  - Ib(t)
  - V2(t)

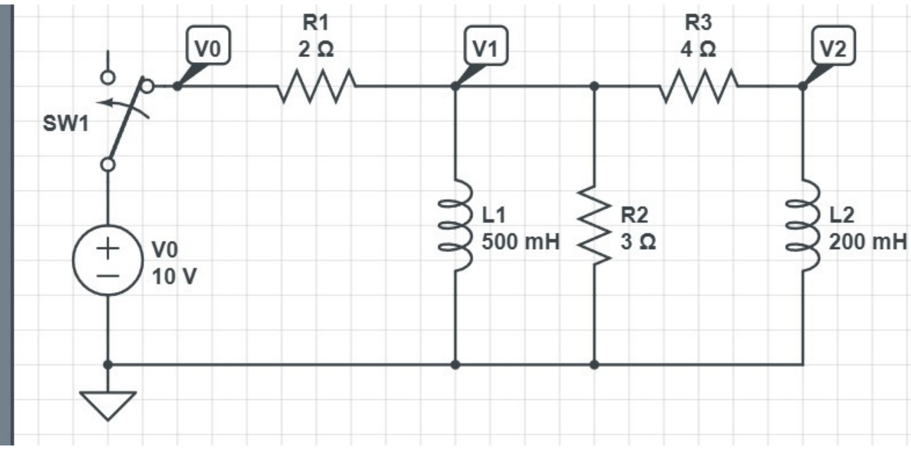


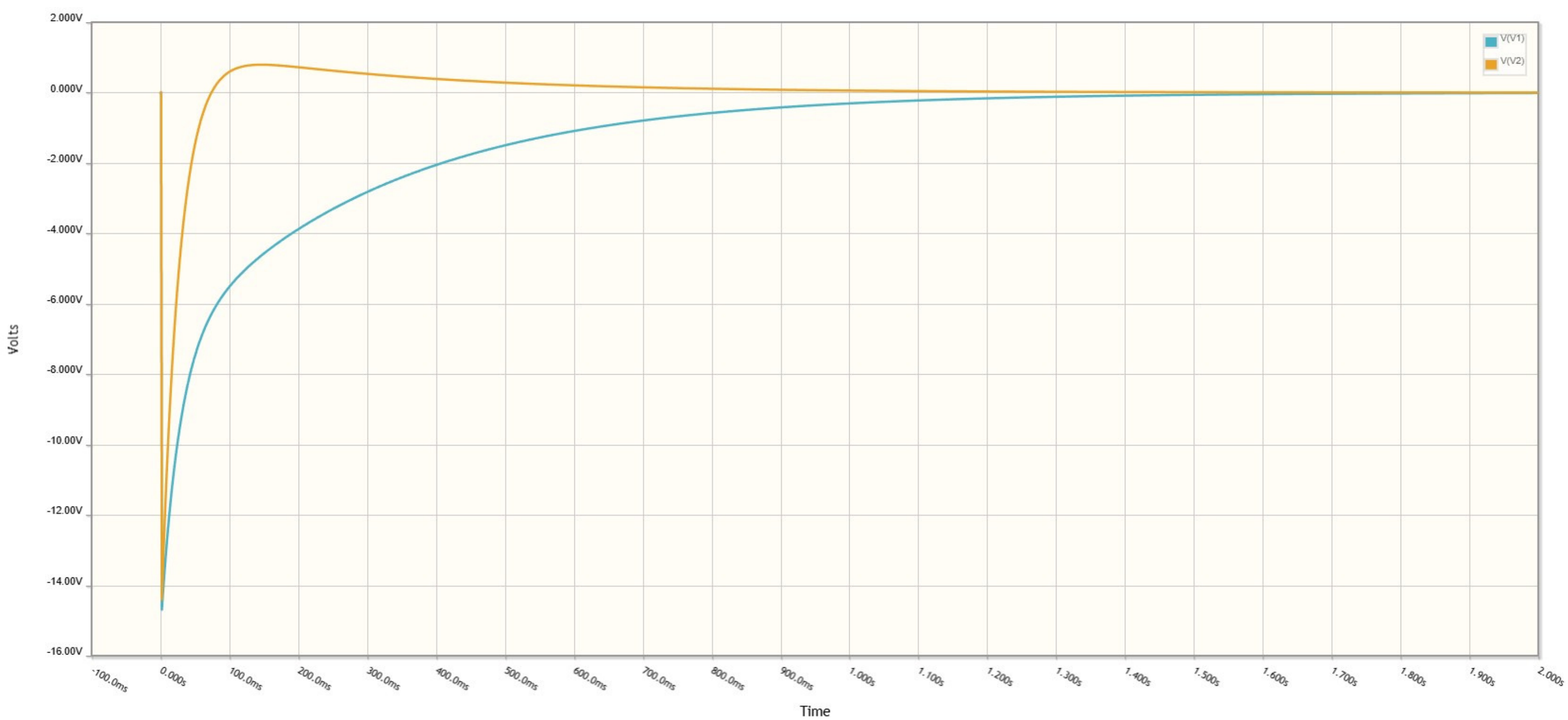
CircuitLab Solution

V2(t) (orange line)

- Jumps to -15V
- Bounces up to +0.8V
- Then goes to zero





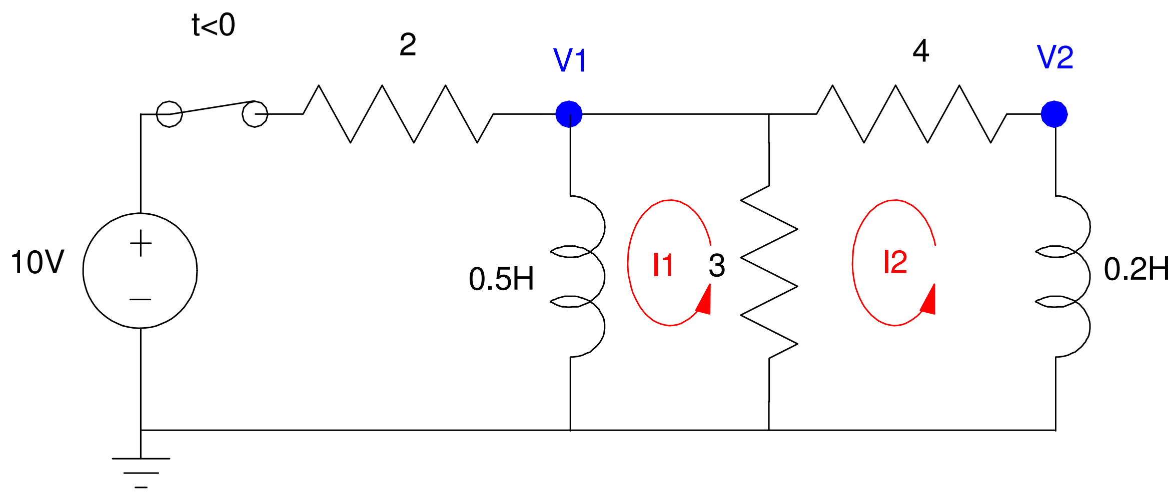


Initial Conditions

- V1(0) = 0
  - At DC, inductors are shorts
- V2(0) = 0

Meaning

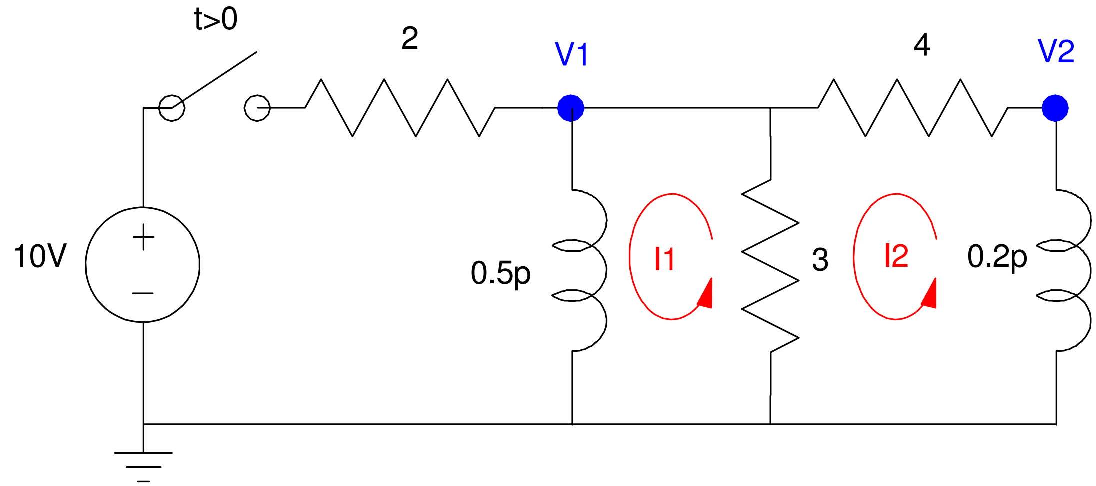
- I1(0) = 5A
- I2(0) = 0A



Differential Equations

Use the p-operator

- L becomes Lp



Write the loop equations

$$0.5p(I_1) + 3(I_1 - I_2) = 0$$

$$3(I_2 - I_1) + 4(I_2) + 0.2p(I_2) = 0$$

or

$$(0.5p + 3)I_1 - (3)I_2 = 0$$

$$-(3)I_1 + (0.2p + 7)I_2 = 0$$

Do some algebra

$$(0.5p+3)I_1-(3)I_2=0$$

$$-(3)I_1+(0.2p+7)I_2=0$$

simplifies to

$$(p^2+41p+120)I_2=0$$

$$(p+3.17)(p+37.12)I_2=0$$

This tells you

$$p=\{-3.17,-37.12\}$$

and that I2(t) is of the form

$$I_2(t)=ae^{-3.17t}+be^{-37.12t}$$

Initial Conditions

- I1(0) = 5
- I2(0) = 0

From the differential equation

$$-(3)I_1 + (0.2p + 7)I_2 = 0$$

you get

$$pI_2 = 15I_1 - 35I_2$$

$$I_2(0) = 0$$

$$pI_2(0) = I_2'(0) = 75$$

Solve for 'a' and 'b'

$$I_2(t) = ae^{-3.17t} + be^{-37.83t}$$

$$I_2(0) = 0 = a + b$$

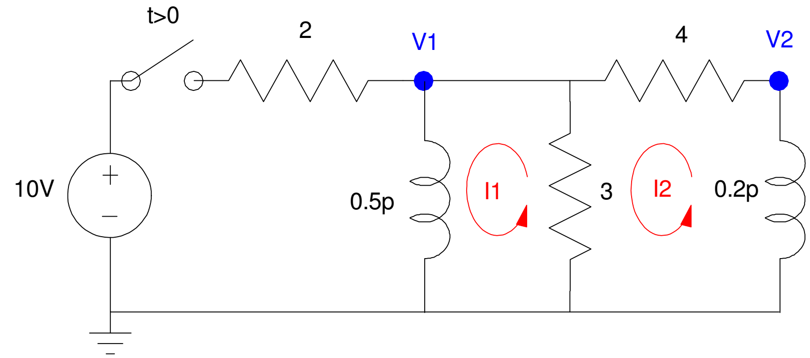
$$I_2'(0) = 75 = -3.17a - 37.12b$$

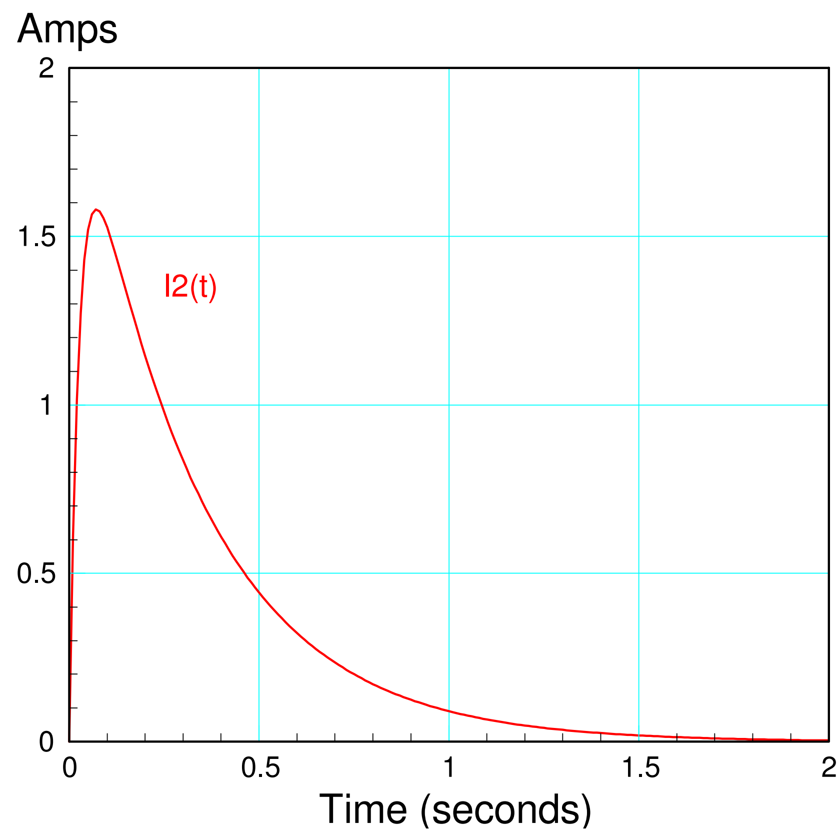
This gives

- a = 2.164
- b = -2.164

or

$$I_2(t) = 2.164e^{-3.17t} - 2.164e^{-37.83t}$$





Finding V2(t)

Note that

$$I_2(t) = 2.164e^{-3.17t} - 2.164e^{-37.83t}$$

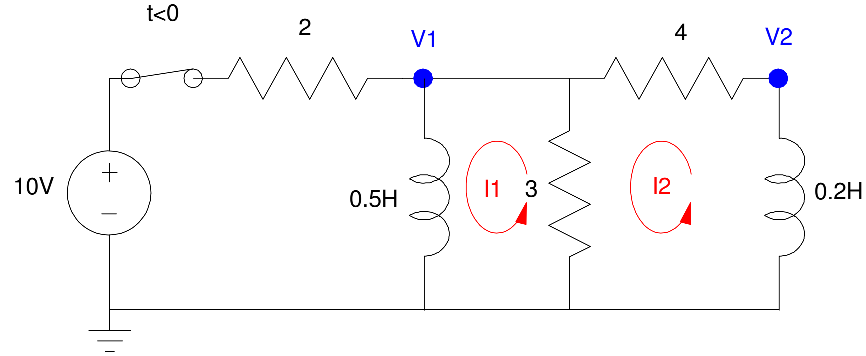
$$V_2(t) = L\frac{dI_2}{dt}$$

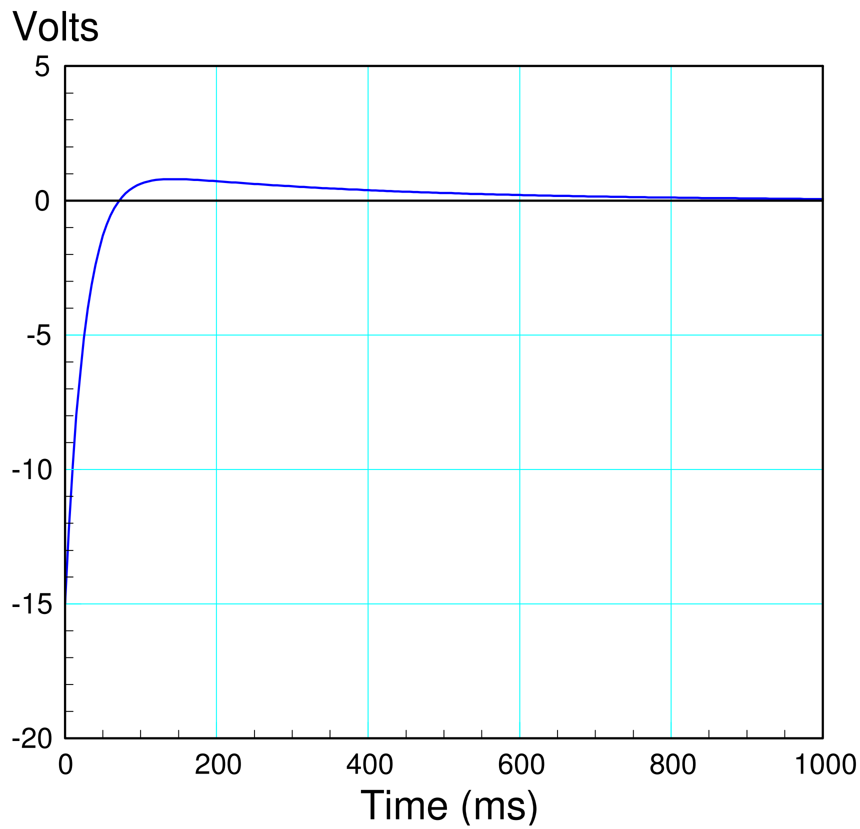
$$L = 0.2H$$

so

$$V_2(t) = -1.373e^{-3.17t} + 16.373e^{-37.12t}$$

This matches CircuitLab



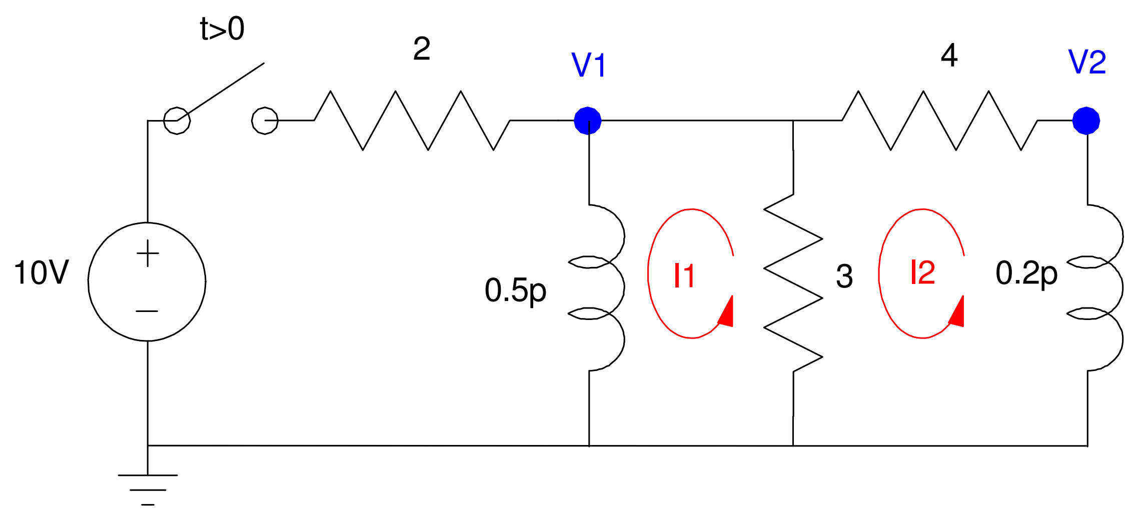


Eigenvalues & Eigenvectors

Assume I2(0) = 1.

What initial condition, I1(0), results in

- I2(t) decaying as fast as possible?
- As slow as possible?
- How fast is that



Also, what is I2(t) if

- I1(0) = I2(0) = 1?

This is an eigenvalue and eigenvector problem

Eigenvalues & Eigenvectors

Recall, the dynamics are

$$(0.5p+3)I_1-(3)I_2=0$$

$$-(3)I_1+(0.2p+7)I_2=0$$

Rewrite this as

$$pI_1=-6I_1+6I_2$$

$$pI_2=15I_1-35I_2$$

In matrix form

$$\begin{bmatrix} pI_1 \\ pI_2 \end{bmatrix}=\begin{bmatrix} -6 & 6 \\ 15 & -35 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$pI=AI$$

Matrix A is 2x2

- It has two eigenvalues
- It has two eigenvectors

In Matlab

```
>> A = [-6,6 ; 15,-35]

    -6     6
    15   -35

>> [M,V] = eig(A);
>> % Eigenvectors
>> M

    0.9046   -0.1853
    0.4263    0.9827

>> % Eigenvalues
>> diag(V)'

   -3.1723  -37.8277
```

Fast Mode

Scale the eigenvectors

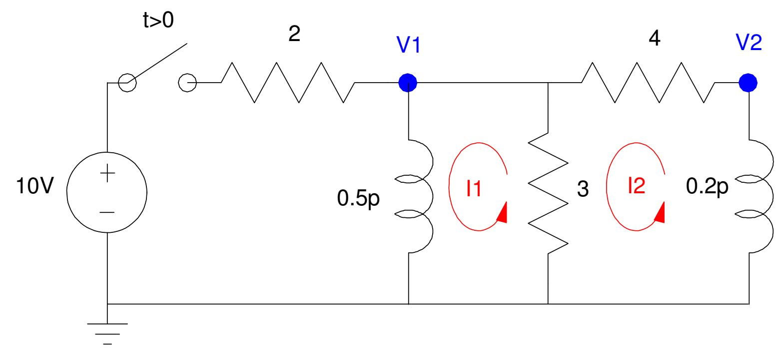
- Make I2's weight 1.000

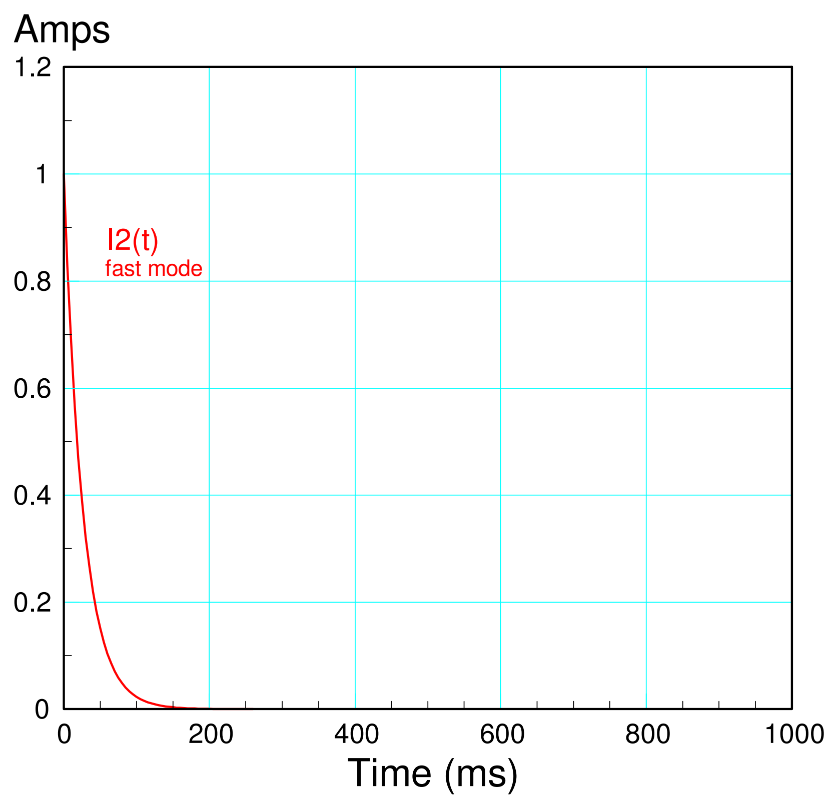
```
>> M(:,1) = M(:,1) / M(2,1);
>> M(:,2) = M(:,2) / M(2,2)
    slow       fast
    2.1218    -0.1885
    1.0000     1.0000
```

Use the fast eigenvector for the I(t=0)

```
>> IC = M(:,2)

I1(0)   -0.1885
I2(0)    1.0000

>> AB = inv(M)*IC

a      0
b      1
```

$$I_2(t) = ae^{-3.17t} + be^{-37.82t}$$





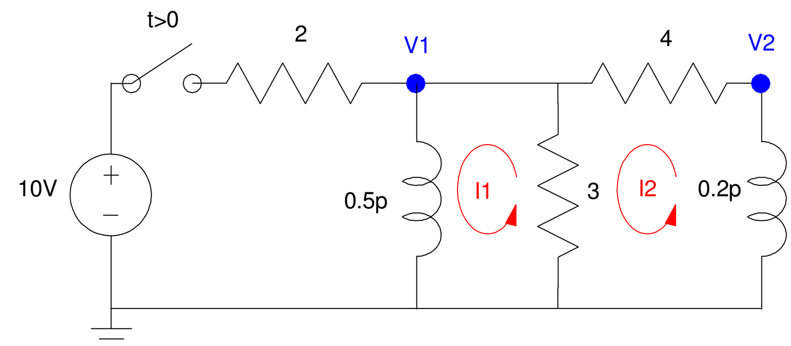
Slow Mode

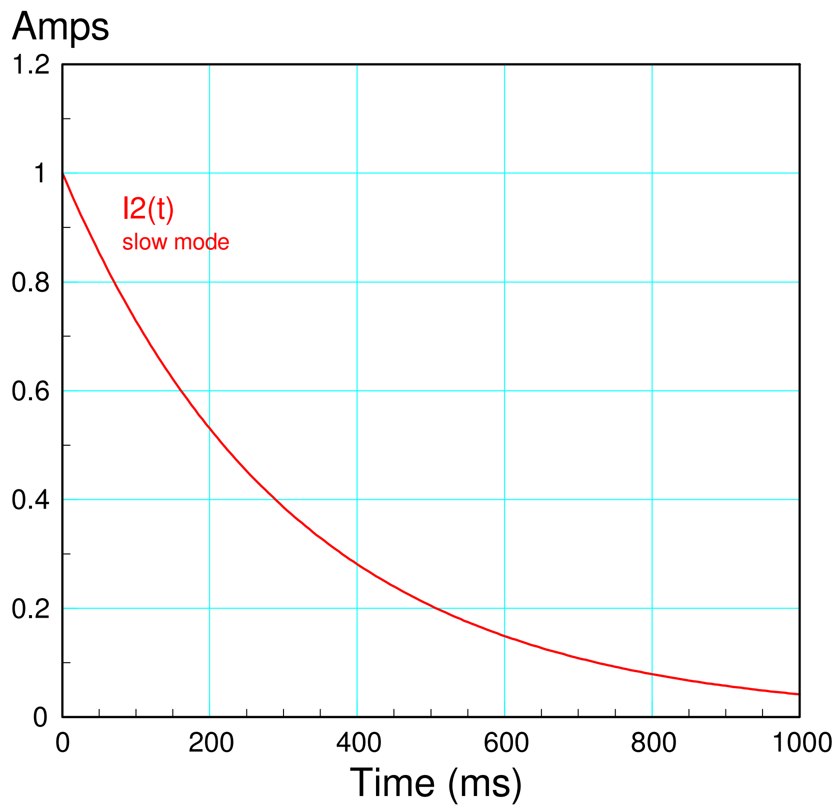
Initial condition is the slow eigenvector

- The transient decays as exp(-3.17t)

```
>> M(:,1) = M(:,1) / M(2,1);
>> M(:,2) = M(:,2) / M(2,2)

    2.1218   -0.1885
    1.0000    1.0000

>> IC = M(:,1)

I1(0)     2.1218
I2(0)     1.0000

>> AB = inv(M)*IC
a      1
b      0
```

$$I_2(t) = ae^{-3.17t} + be^{-37.82t}$$





Other Initial Conditions

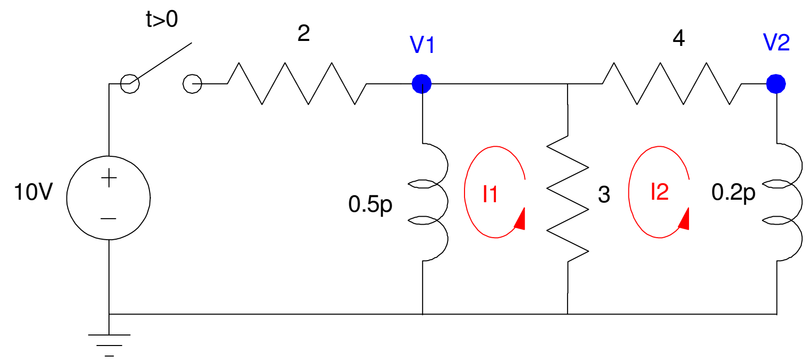
- I1(0) = I2(0) = 1

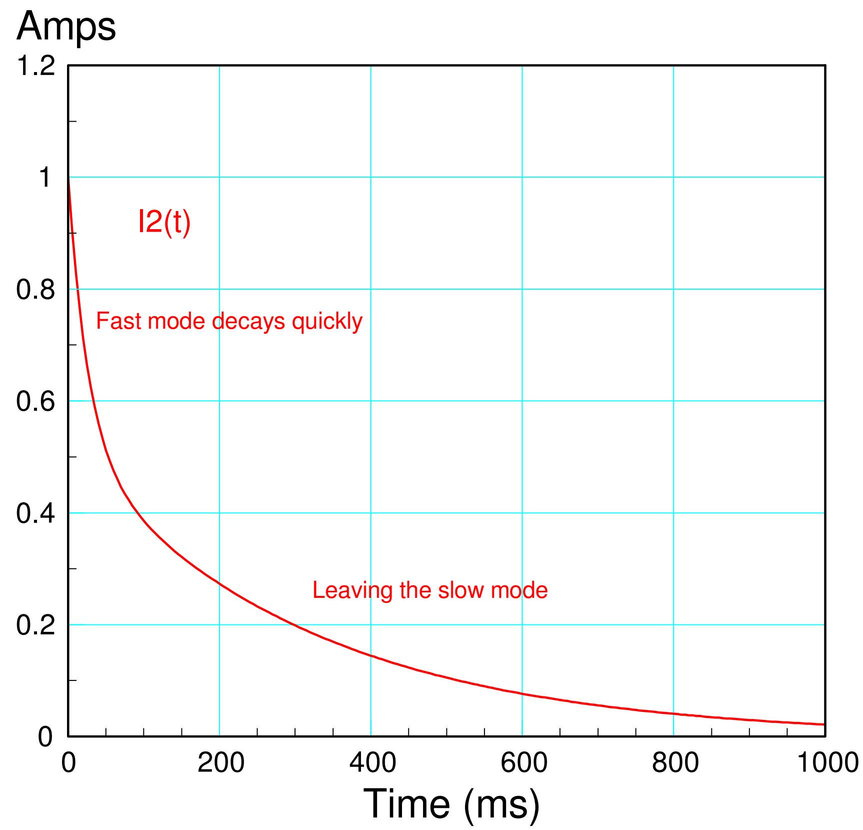
```
>> IC = [1;1]

I1(0)      1
I2(0)      1

>> AB = inv(M)*IC

a     0.5144
b     0.4856
```

$$I_2(t) = ae^{-3.17t} + be^{-37.82t}$$

Note

- Both modes are excited
- The fast mode decays quickly
- Leaving the slow mode





Summary

The natural response of an RL circuit is the response when

- The inductors have an initial current and
- The input turns off.

The natural response can be found using

- The differential equation that describe the circuit
  - p-operators are useful here
- The initial conditions at t=0, or
- The circuit's eigenvalues and eigenvectors