Natural Response of RC Circuits

ECE 211 Circuits I

Lecture #35

Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

Previous Lectures

Steady-state response of R circuits with constant inputs

- DC Analysis

Steady-state response of RC circuits with sinusoidal inputs

- Phasor Analysis

Time-domain simulations in Matlab and CircuitLab

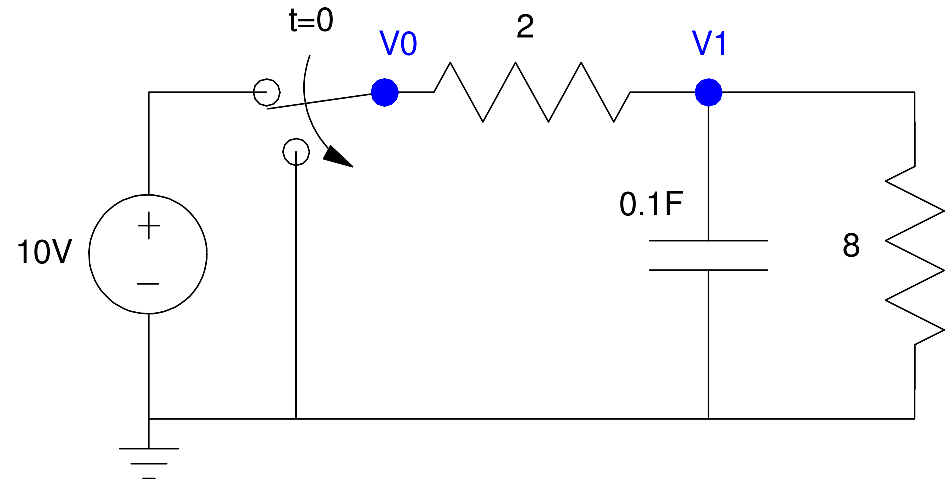
- RLC circuits
- Find the differential equations that describe the circuit
- Solve using numeric integration in Matlab
- Solve using Time-Domain simulations in CircuitLab

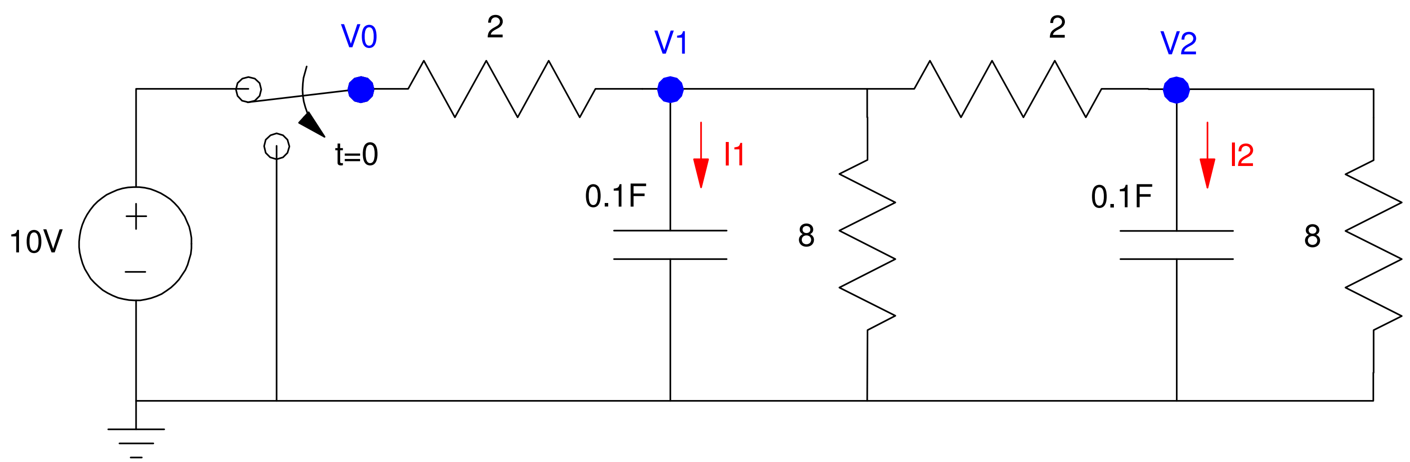
This lecture

- Natural Response of RC circuits
  - Initial condition at t=0
  - Input removed for t>0

Two cases

- 1st-order RC circuit
- 2nd-order RC circuit



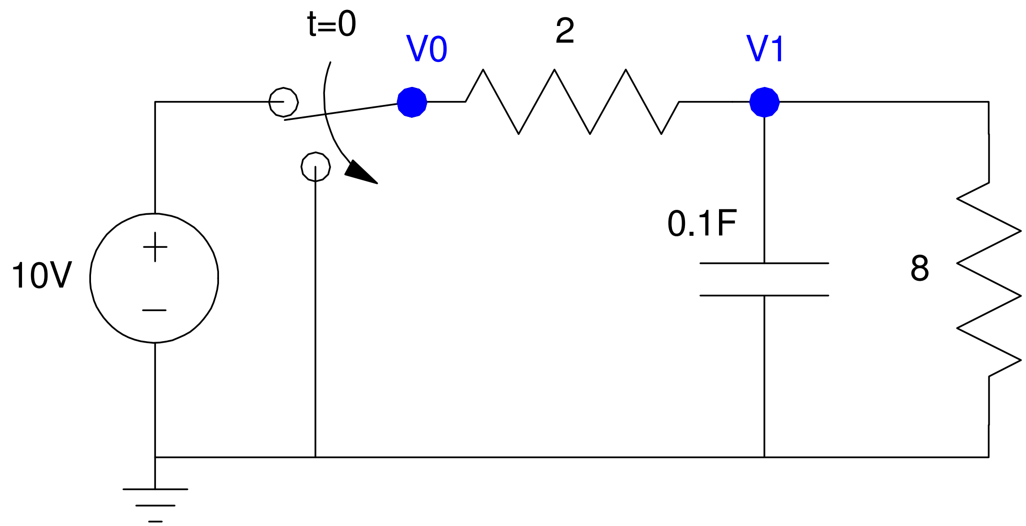


Example 1: 1-Stage RC Circuit

Switch opens at t=0

- V0 = 10V for t < 0
- V0 = 0V for t > 0

Find V1(t) for t > 0

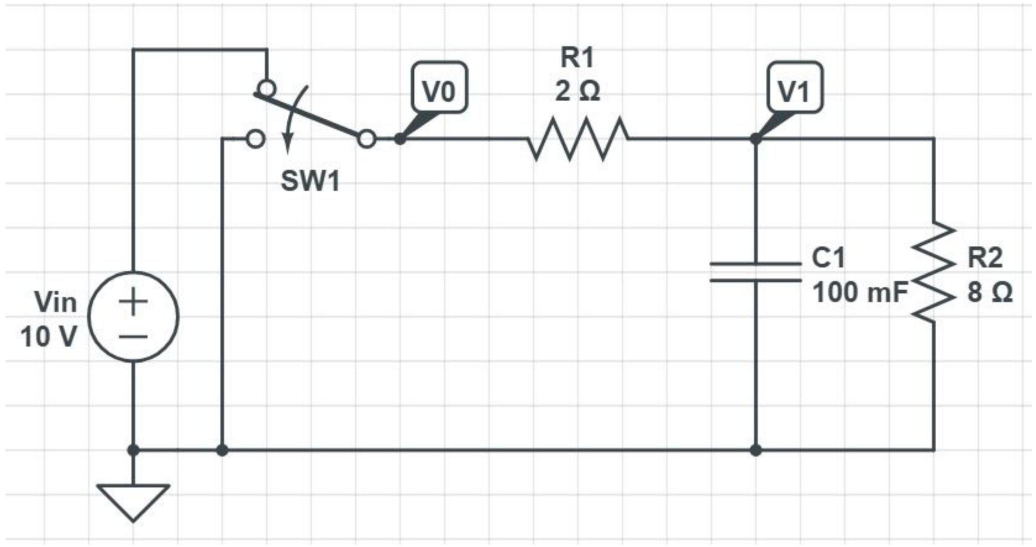


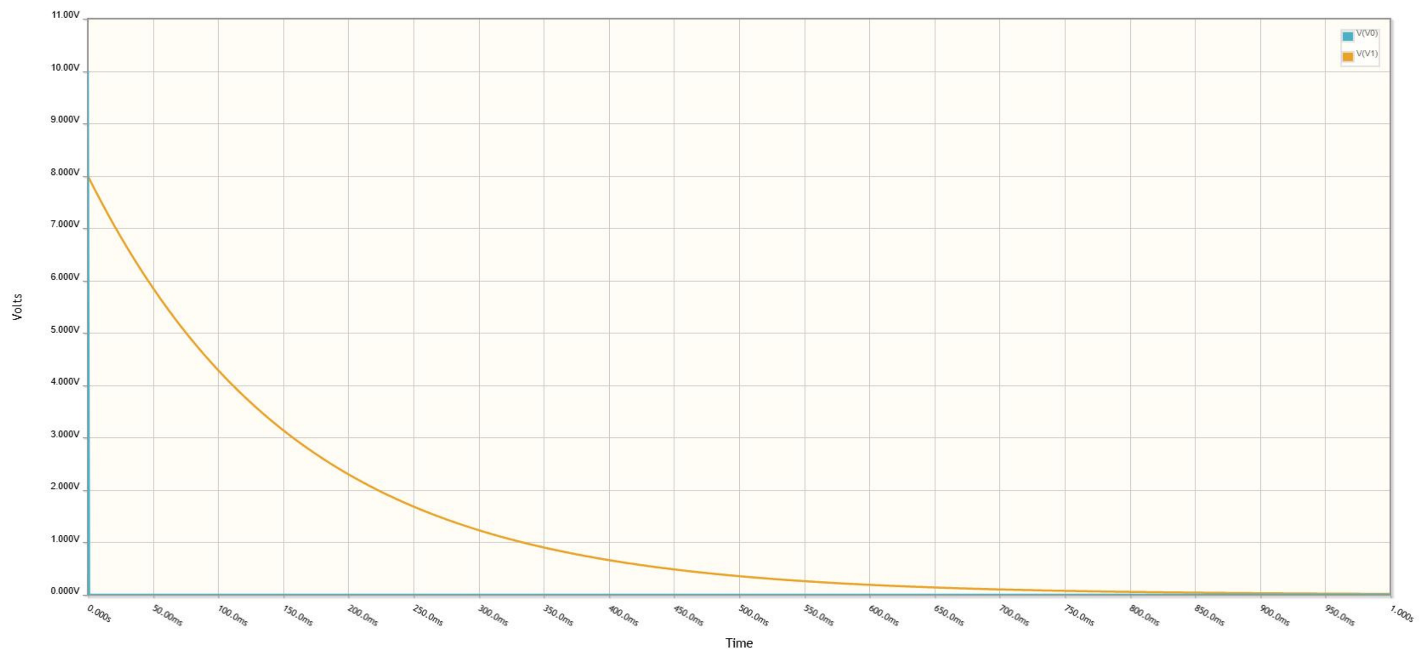
CircuitLab Solution

- Option #1
- Use a switch at t=0
- Run a time-domain simulation

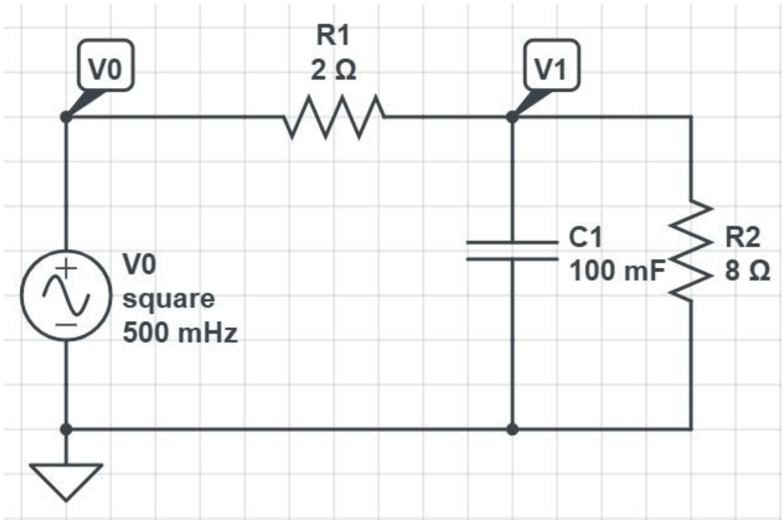
Note

- V1(t) starts at 8V
- Decays to 0V
- In about 700ms





CircuitLab Solution

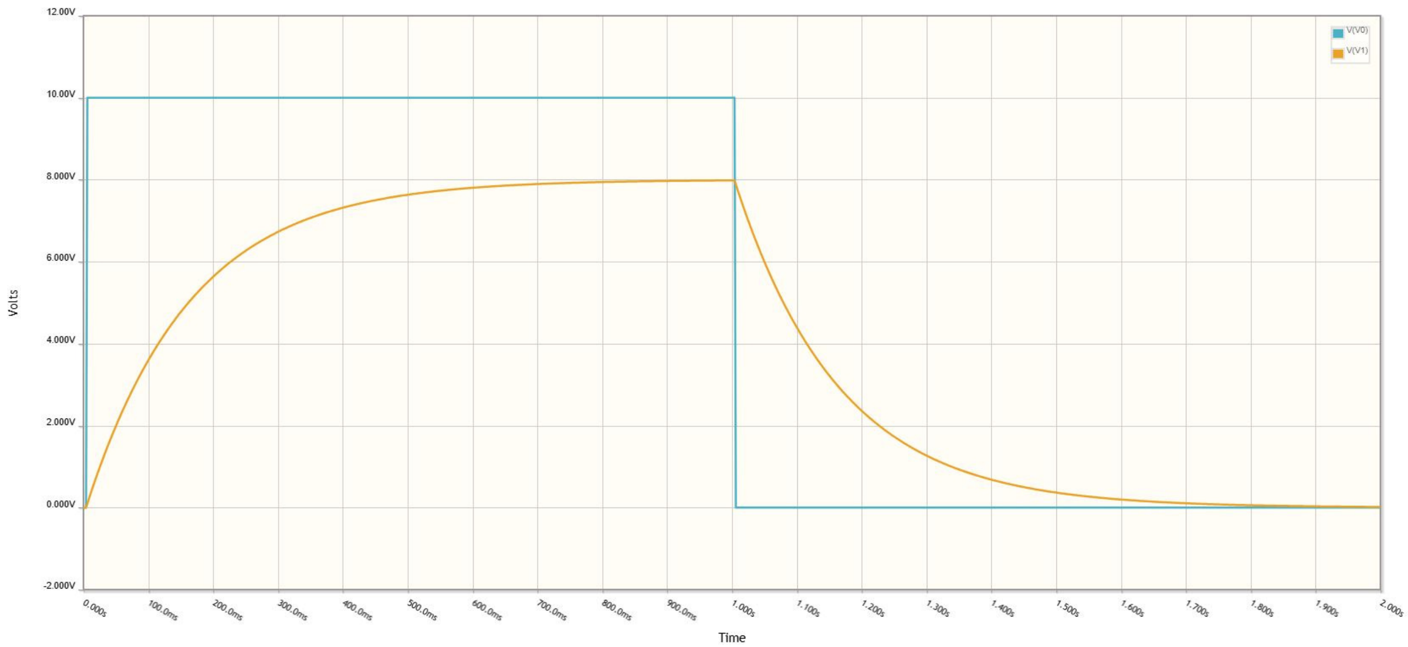


Option #2: Using a periodic input

- 0.5Hz Square wave5V amplitude
  - Long enough that V1 reaches steady-state
- 5V offset
- 0.1 degree phase shift

Result: Both

- Step response
  - V0 = 10
- Natural response
  - V0 = 0



Differential Equations for t>0

Using the p-operator

$$I_1 = C\frac{dV_1}{dt} = CpV_1$$

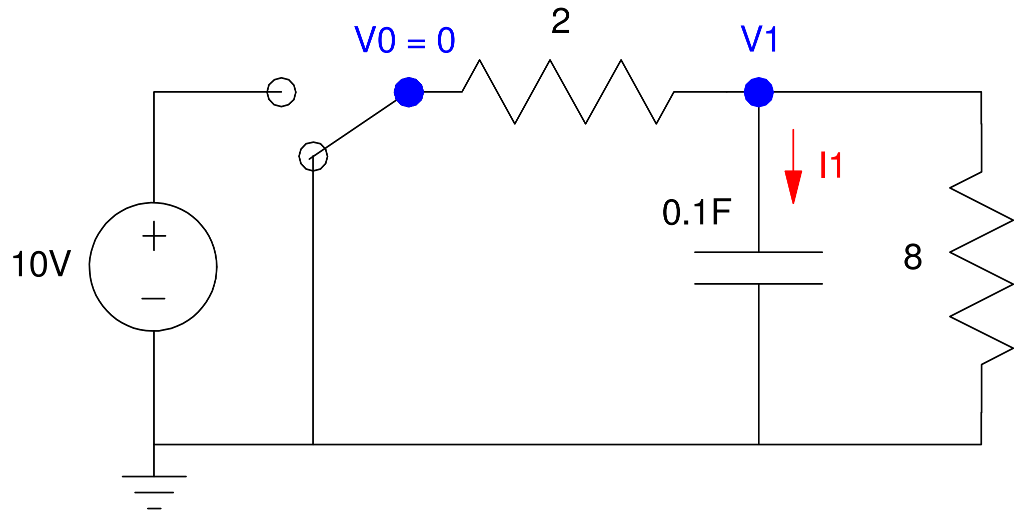
$$Z_c = \frac{V_1}{I_1} = \frac{1}{Cp} = \frac{10}{p}$$

Node equation at V1

$$\left(\frac{V_1-0}{2}\right) + \left(\frac{V_1}{10/p}\right) + \left(\frac{V_1}{8}\right) = 0$$

Simplify

$$(p+6.25)V_1 = 0$$



Initial Conditions

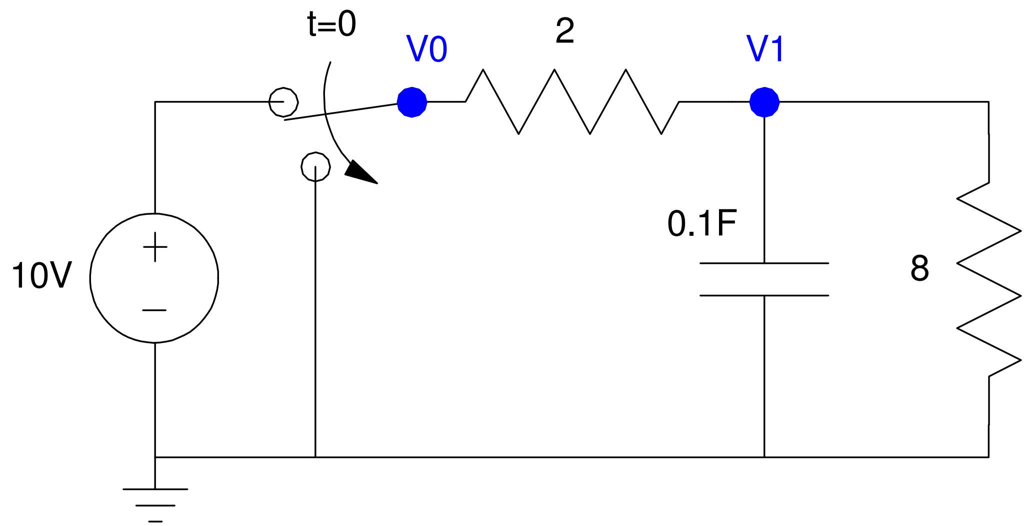
Just prior to t=0

- V0 = 10

By voltage division

$$V_1(0) = \left(\frac{8}{8+2}\right)10$$

$$V_1(0) = 8$$



Natural Response

$$(p + 6.25)V_1 = 0$$

$$V_1(0) = 8$$

p = -6.25

$$V_1(t) = a \cdot e^{pt}$$

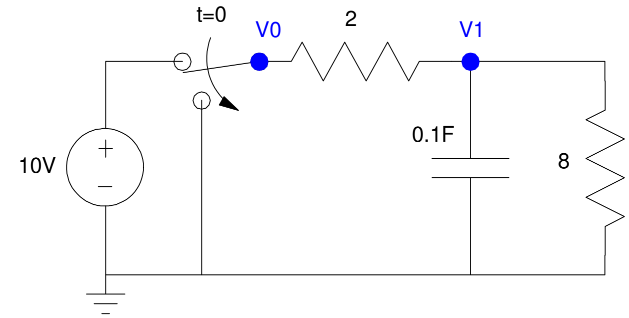
$$V_1(t) = a \cdot e^{-6.25t}$$

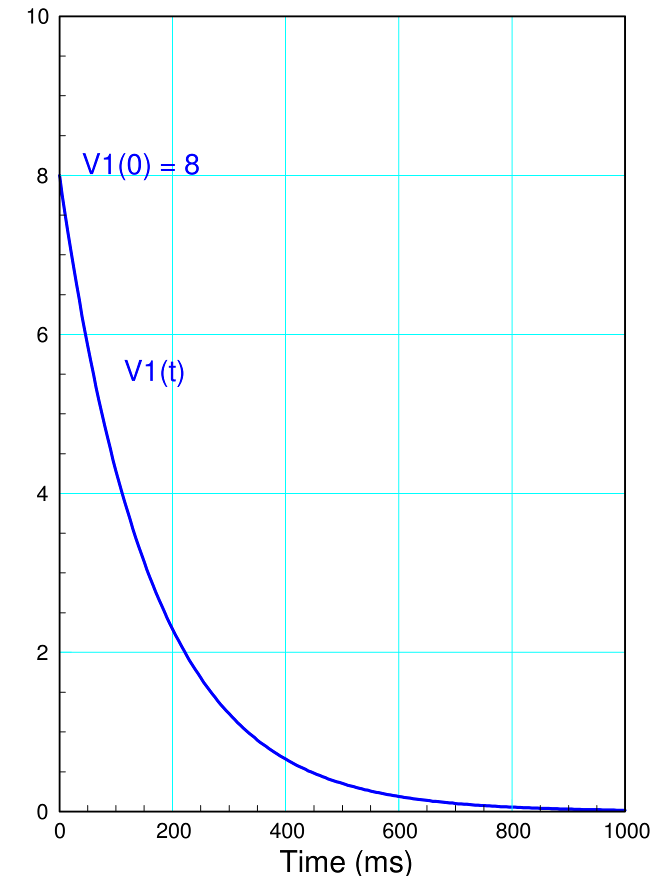
Plug in the initial condition

$$a = 8$$

$$V_1(t) = 8e^{-6.25t}$$

This is the equation for V1(t)

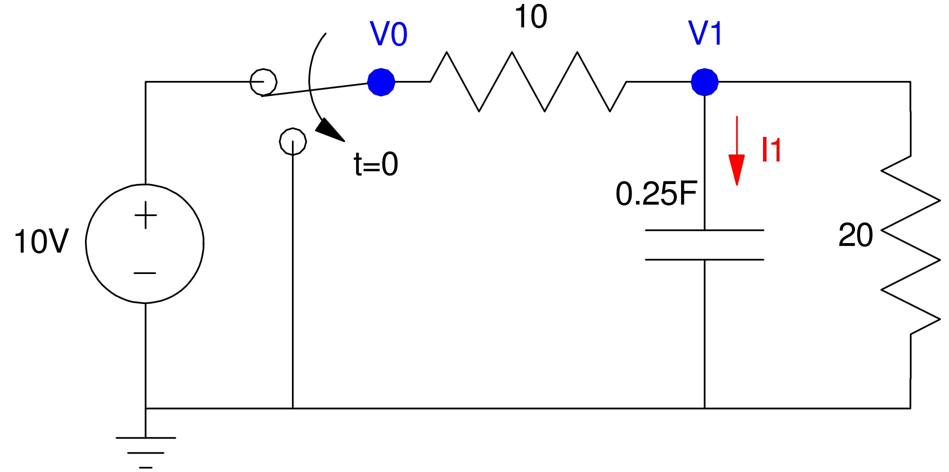




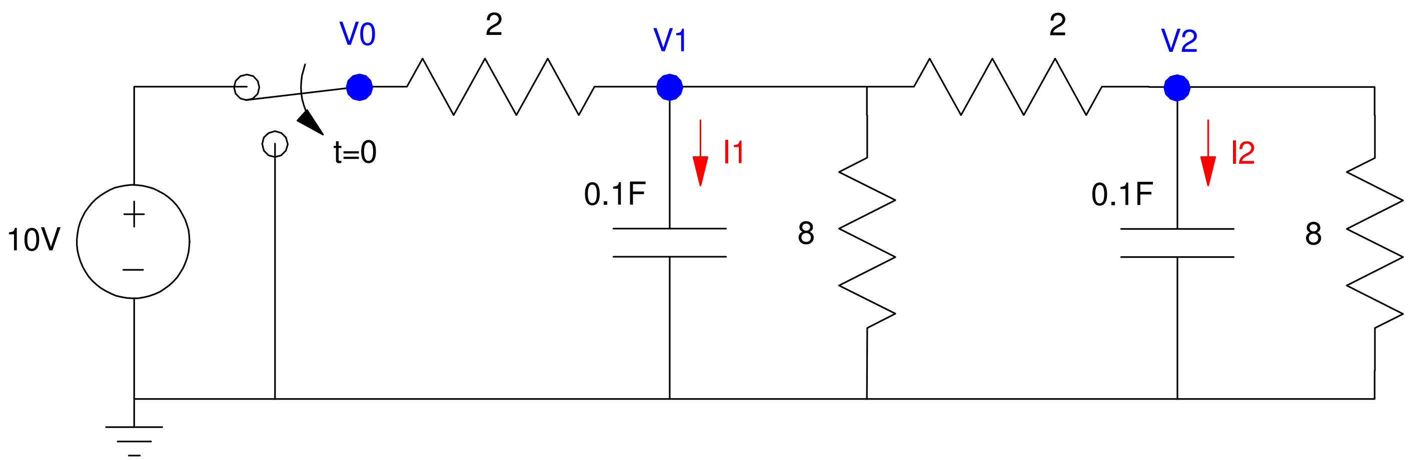
Practice Problem

Find the intial condition, V1(0)

Find the differential equation for V1

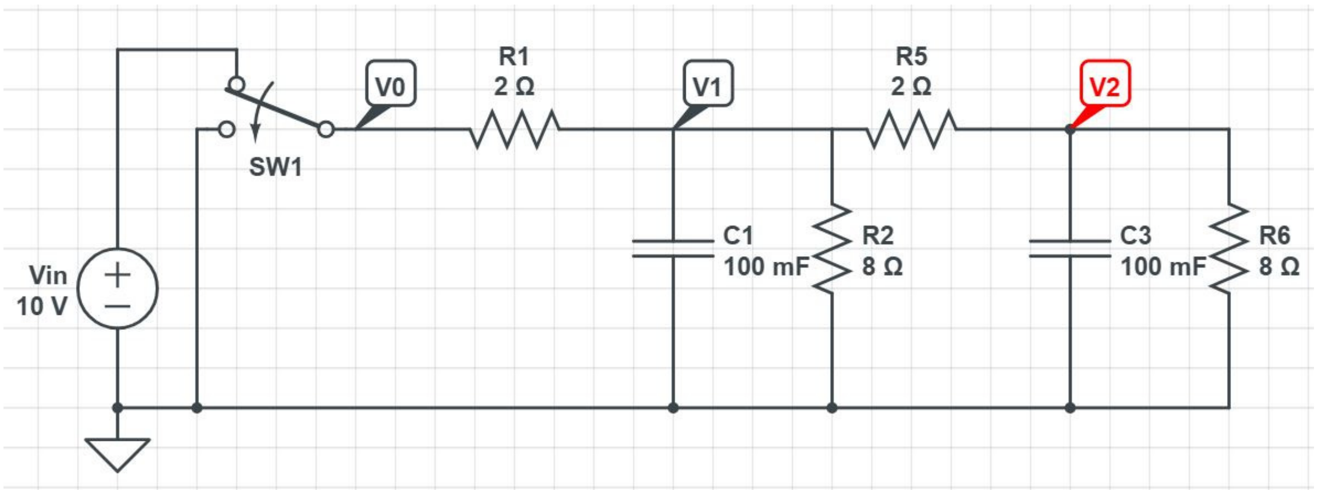


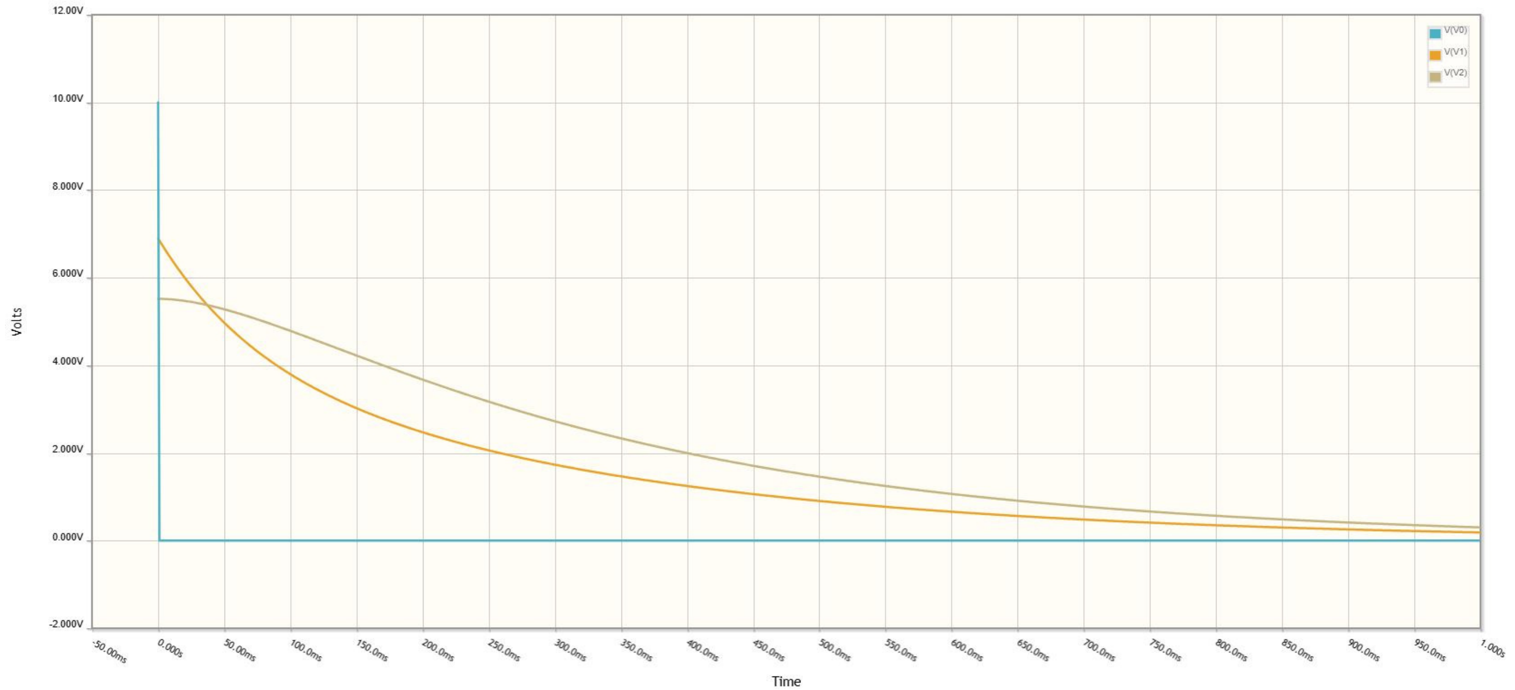
Example 2: 2-Stage RC Circuit



CircuitLab Solution

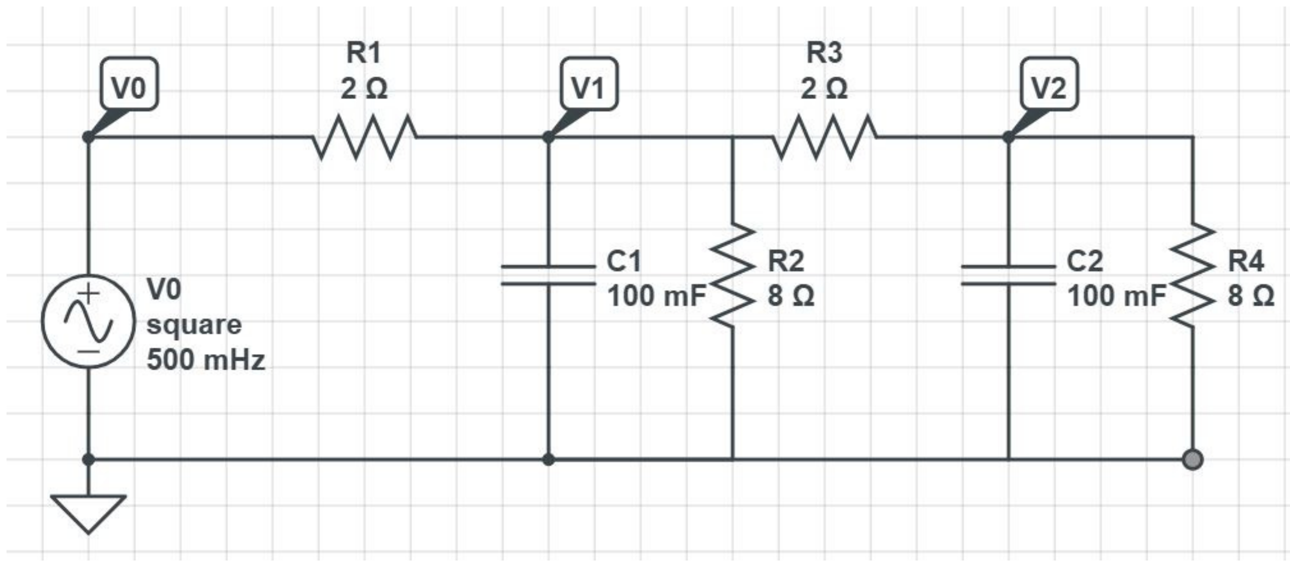
Version 1

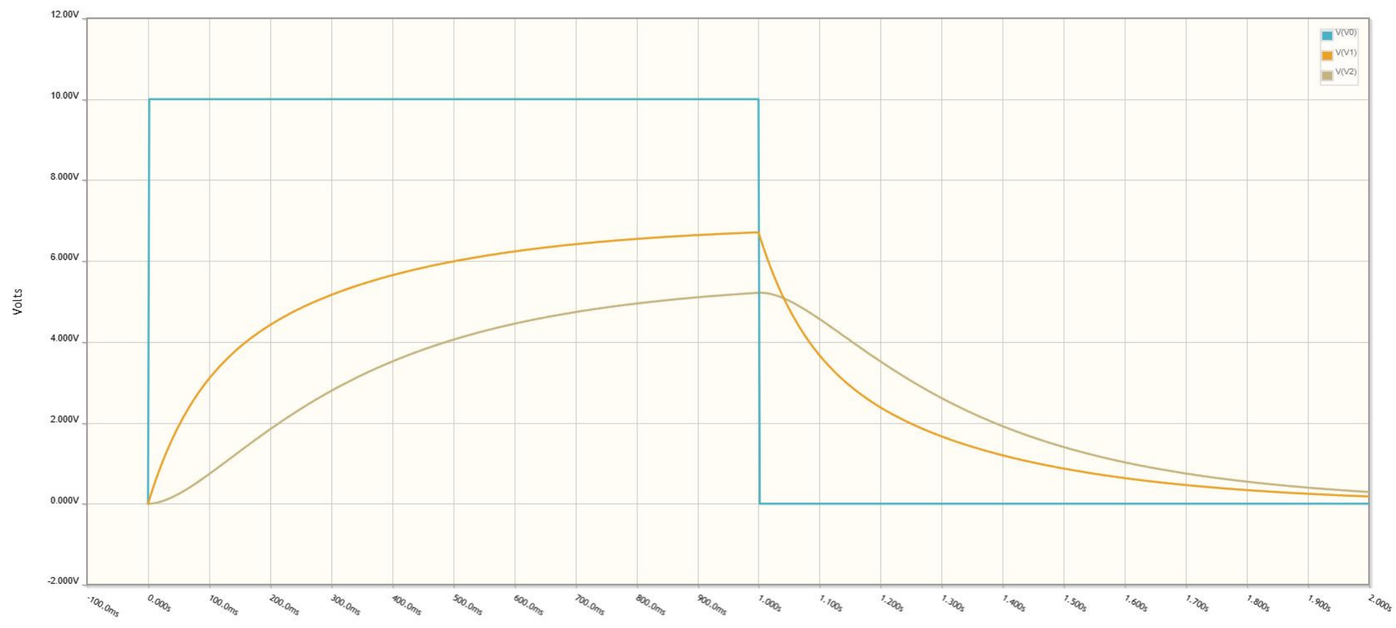




CircuitLab Solution

Version 2



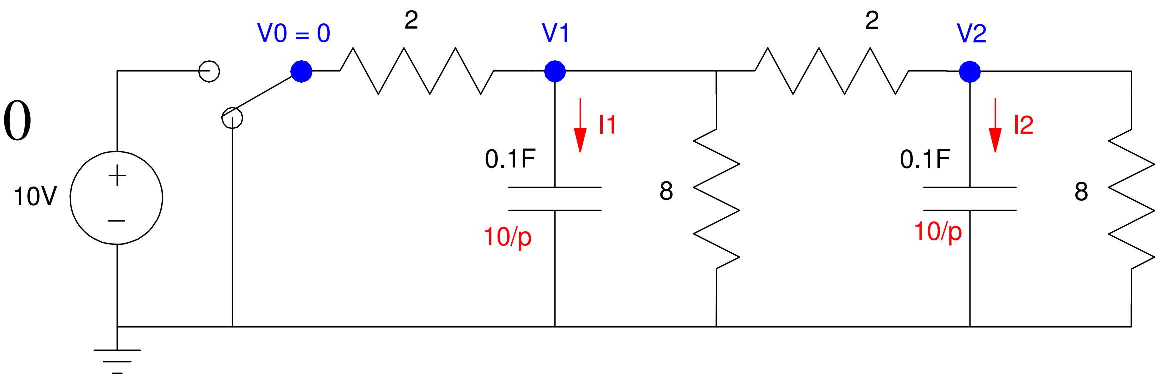


Differential Equations

Voltage node equations

$$\left(\frac{V_1}{2}\right) + \left(\frac{V_1}{10/p}\right) + \left(\frac{V_1}{8}\right) + \left(\frac{V_1 - V_2}{2}\right) = 0$$

$$\left(\frac{V_2 - V_1}{2}\right) + \left(\frac{V_2}{10/p}\right) + \left(\frac{V_2}{8}\right) = 0$$



Do some algebra

$$(p + 11.25)V_1 - (5)V_2 = 0$$

$$-(5)V_1 + (p + 6.25)V_2 = 0$$

More algebra

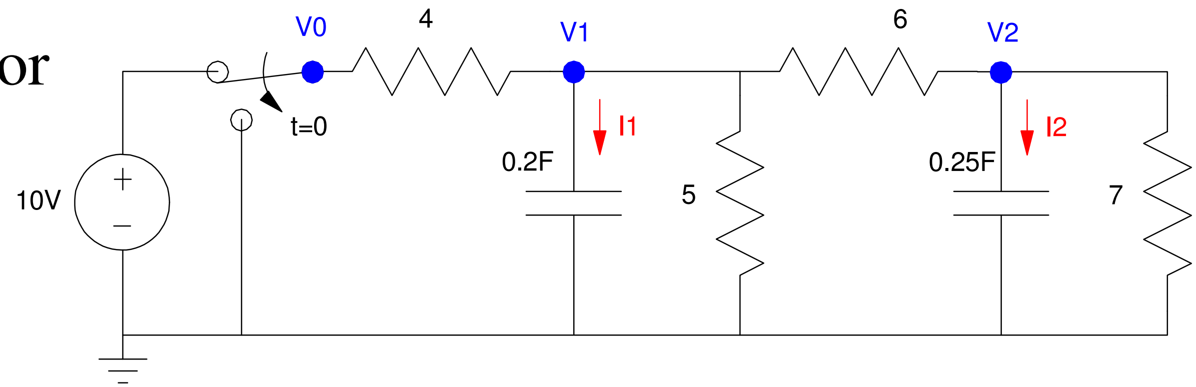
$$(p^2 + 18.5p + 45.3125)V_2 = 0$$

$$(p + 15.7644)(p + 2.7356)V_2 = 0$$

Practice Problem

Find the differential equations for

- V1'
- V2'



Practice Problem

Find the differential equation for V2

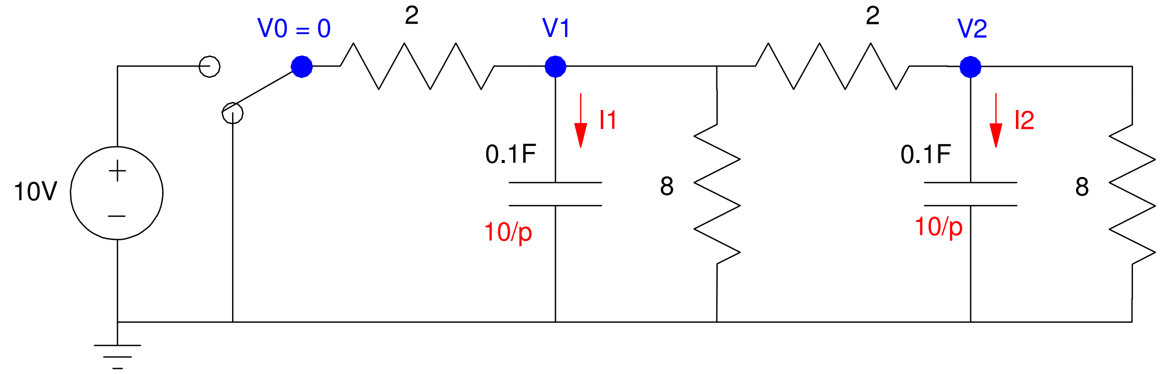
$$(p+3.0833)V_1 - 0.8333V_2 = 0$$

$$(p+1.2381)V_2 - 0.6667V_1 = 0$$

Initial Conditions

From voltage node equations

- V0 = 10.000
- V1 = 6.897
- V2 = 5.517



Meaning

$$V_2(0) = 5.517$$

To find V2'(0), use the differential equations

$$-(5)V_1 + (p + 6.25)V_2 = 0$$

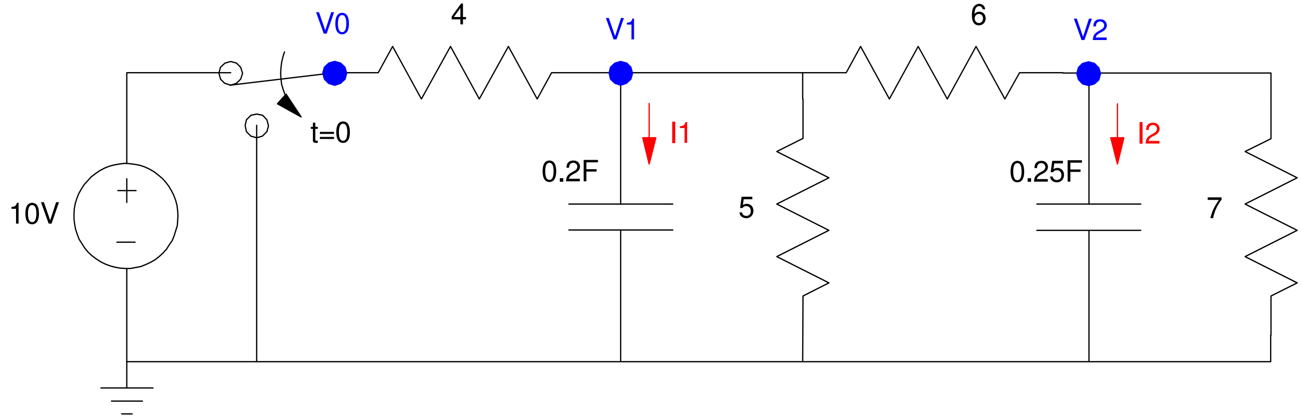
$$pV_2 = 5V_1 - 6.25V_2 = 0$$

$$V_2'(0) = 0$$

Practice Problem

Find the initial conditions

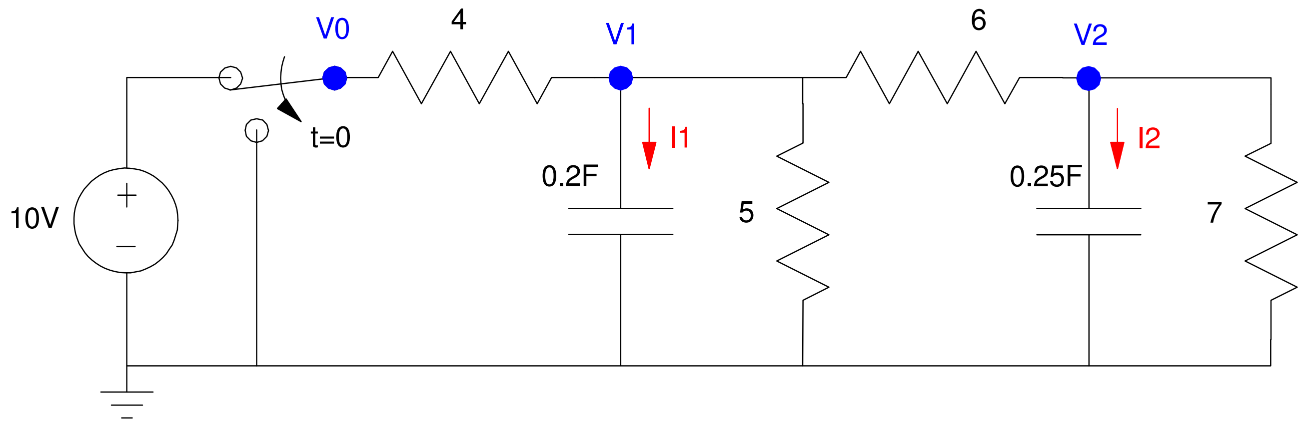
- V1(0)
- V2(0)



Practice Problem

Find the initial conditions

- V1'(0)
- V2'(0)



Assume

$$pV_1 = -3.0833V_1 + 0.8333V_2$$

$$pV_2 = 0.6667V_1 - 1.2381V_2$$

Also

V1(0) = 5.000V

V2(0) = 2.917V

Natural Response for V2(t)

$$(p+15.7644)(p+2.7356)V_2 = 0$$

Meaning

$$V_2(t) = ae^{-2.7356t} + be^{-15.7644t}$$

Plug in the initial conditions

$$V_2(0) = 5.517 = a + b$$

$$V_2'(0) = 0 = -2.7356a - 15.7644b$$

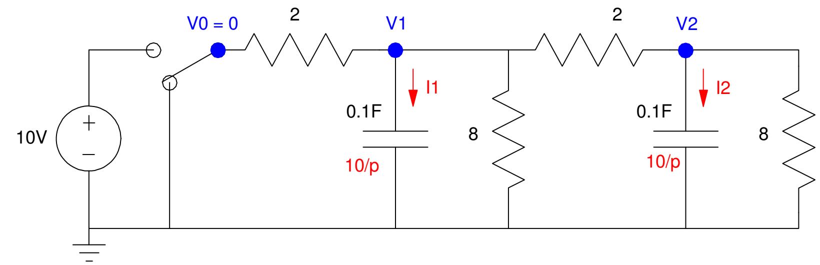
Solving

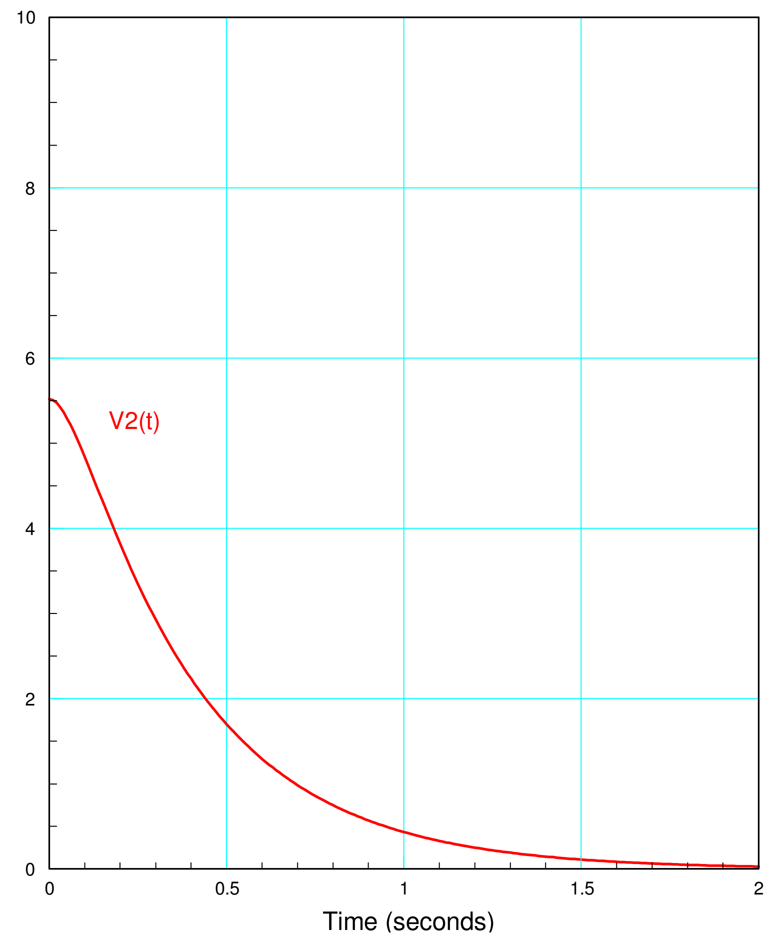
$$a = 6.6754$$

$$b = -1.1584$$

and

$$V_2(t) = 6.6754e^{-2.7356t} - 1.1584e^{-15.7644t}$$





Natural Response for V1(t)

The energy dissiptes the same at V1

$$V_1(t) = ae^{-2.7356t} + be^{-15.7644t}$$

Intitial Conditions:

$$V_1(0) = 6.897V$$

$$(p + 11.25)V_1 - (5)V_2 = 0$$

$$pV_1 = -11.25V_1 + 5V_2$$

$$pV_1(0) = -50$$

Plug in the initial conditions

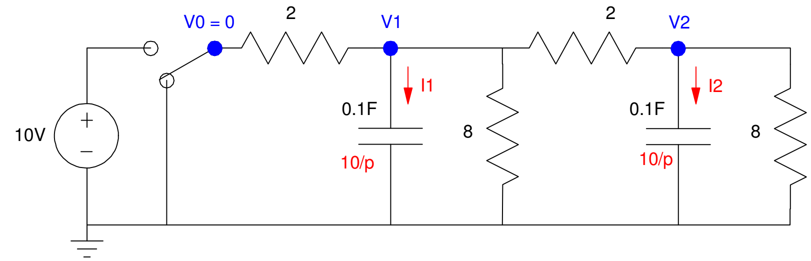
$$V_1(0) = 6.897 = a + b$$

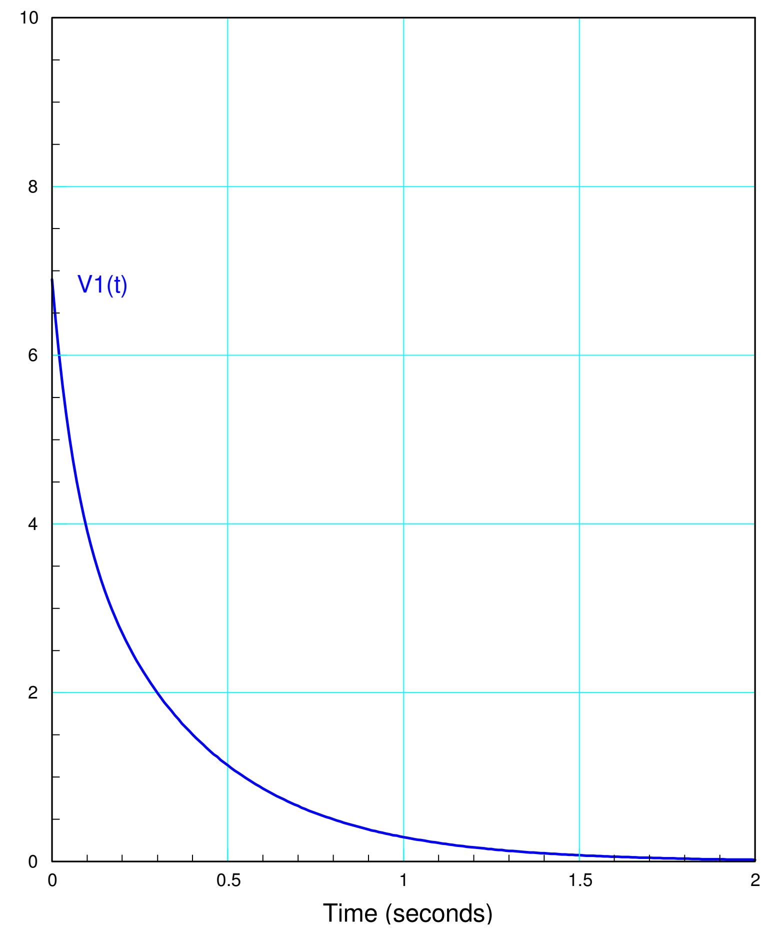
$$V_1'(0) = -50 = -2.7356a - 15.7644b$$

Gives

$$a = 4.5075$$

$$b = 2.3895$$





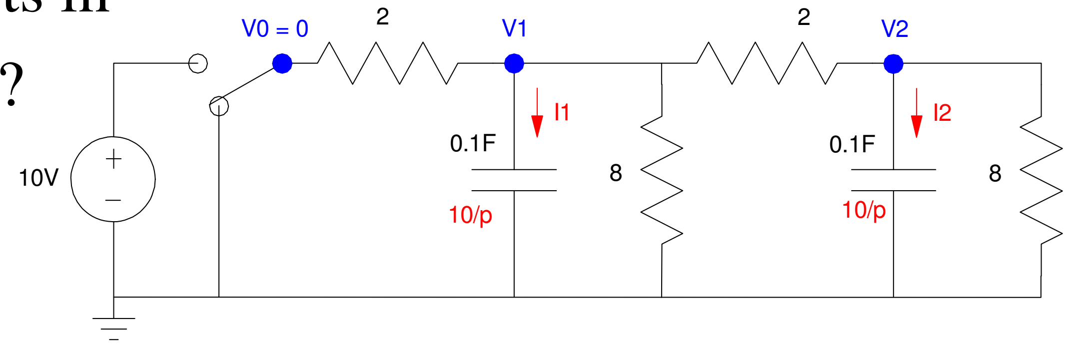
Eigenvalues & Eigenvectors

Assume V2(0) = 10V

What initial condition of V1(0) results in

- V2(t) decaying as fast as possible?
- As slow as possible?
- How fast does V1(t) decay?

What happens if V1(0) = 1?



This is an eigenvalue / eigenvector problem

Eigenvalues & Eigenvectors (cont'd)

First, write the differential equations

$$(p+11.25)V_1-(5)V_2=0$$

$$-(5)V_1+(p+6.25)V_2=0$$

Solve for the derivatives

$$pV_1=-11.25V_1+5V_2$$

$$pV_2=5V_1-6.25V_2$$

Place in matrix form

$$\begin{bmatrix} pV_1 \\ pV_2 \end{bmatrix}=\begin{bmatrix} -11.25 & 5 \\ 5 & -6.25 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$pV=AV$$

In Matlab,

Find the eigenvalues and eigenvectors

This is a second-order system

- Two capacitors

It has two eigenvalues

- *how* the system behaves

It has two eigenvectors

- *what* behaves that way

Matlab Code

```
>> A = [-11.25, 5 ; 5, -6.25]

  -11.2500    5.0000
    5.0000   -6.2500

>> [M,N] = eig(A)

eigenvectors
V1   -0.8507   -0.5257
V2    0.5257   -0.8507

eigenvalues
    -14.3402   -3.1598
```

Fast Mode

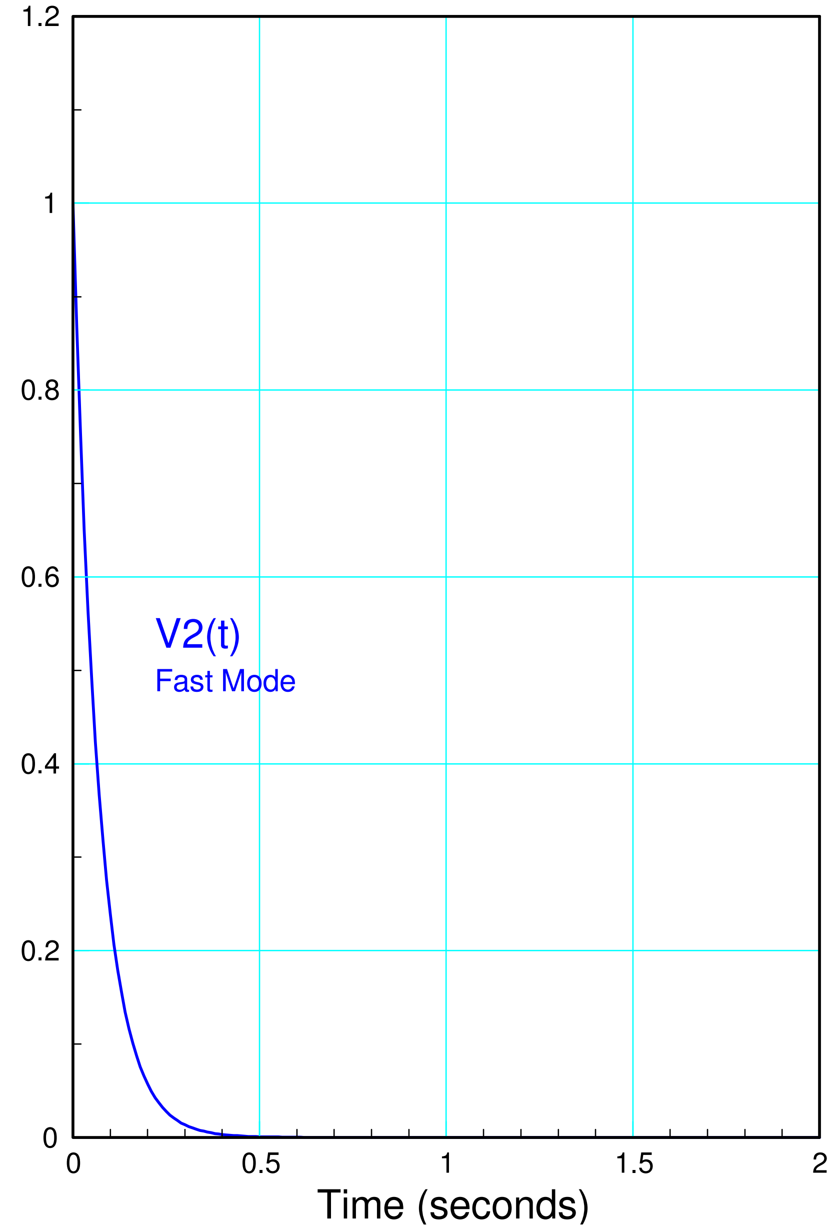
Make the initial conidtion the fast mode

- The system will decay as exp(-14.34t)

```
>> V0 = M(:,1)

   -0.8507
    0.5257

>> V0 = V0 / V0(2)
```

V1   -1.6180
V2   1.0000

```
>> N(1,1)

  -14.3402
```



Slow Mode

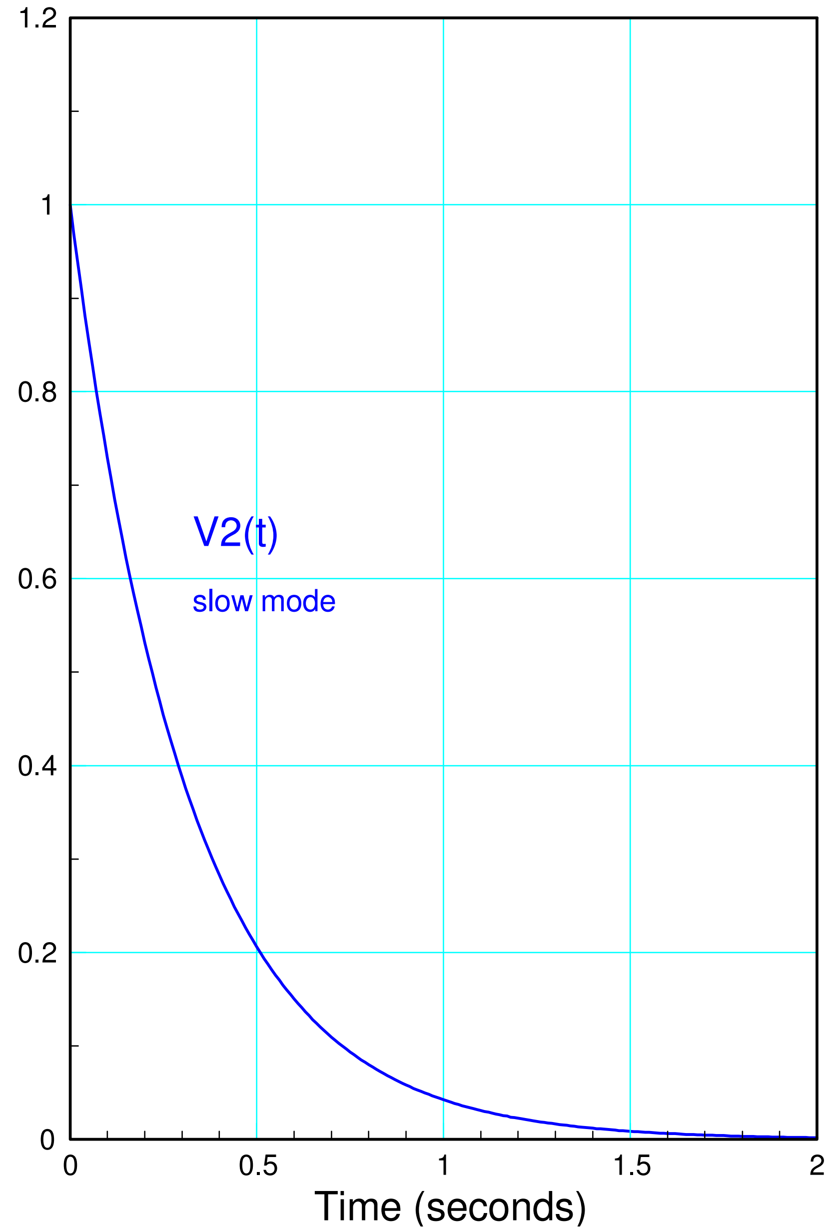
Make the initial condition the slow mode

- The system will decay as exp(-3.15t)

```
>> V0 = M(:,2)

   -0.5257
   -0.8507

>> V0 = V0 / V0(2)
```

V1   0.6180
V2   1.0000

```
>> N(2,2)

   -3.1598
```



What if V1(0)=1?

The eigenvectors & eigenvalues determine V(t)

$$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = a\begin{bmatrix} -0.8507 \\ 0.5257 \end{bmatrix} e^{-14.34t} + b\begin{bmatrix} -0.5257 \\ -0.8507 \end{bmatrix} e^{-3.16t}$$

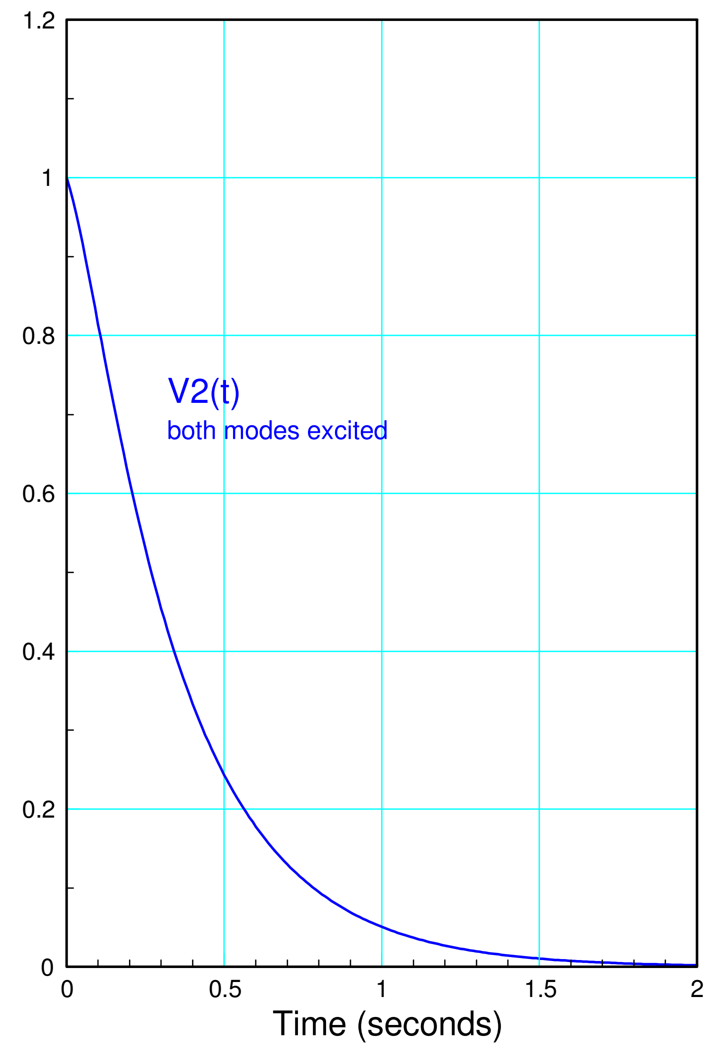
If

- V1(0) = V2(0) = 1

'a' and 'b' are found using the eigenvetors (M)

```
>> IC = [1;1]

     1
     1

>> inv(M)*IC

a   -0.3249
b   -1.3764
```

$$v_2(t) = a \cdot 0.5257 \cdot e^{-14.34t} + b \cdot (-0.8507) \cdot e^{-3.16t}$$



Summary

The natural response of an RC filter is the response when

- The capacitors have an initial voltage and
- The input turns off.

The natural response can be found using

- The differential equation that describe the circuit
  - p-operators are useful here
- The initial conditions at t=0