The p-Operator

ECE 211 Lecture #34

Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

Background

Differential equations describe circuits with inductors and/or capacitors

- $V = L\frac{dI}{dt}$
- $I = C\frac{dV}{dt}$

Likewise, the solution to differential equations is fundamental to circuit analysis.

Transient vs. Stady-State

If you have

- A dynamic system
- With a constant input
- That changes

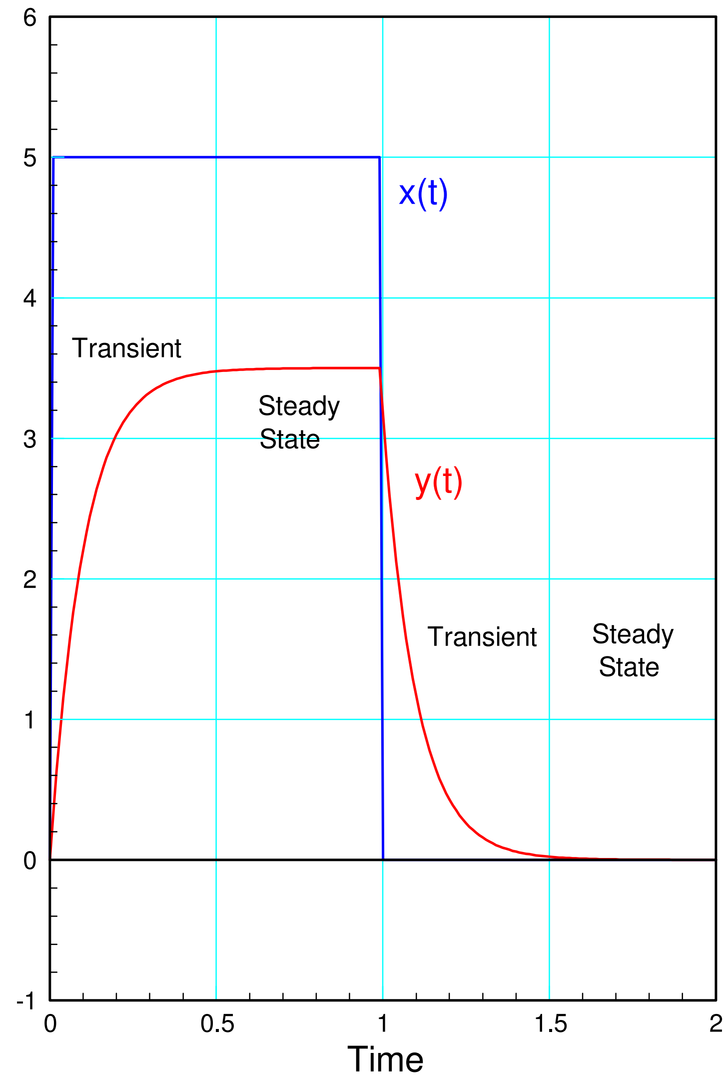
The output will have

- A transient response,
- Followed by a steady-state response

So far, we've looked at finding the steady-state solution

- How do you find the transient response?

$$y' + 10y = 7x$$



Tools for finding the transient response

There are several tools available

- Calculus I - III:
  - Guess the form of the answer
  - Solve for constants to make the equation balance
- CircuitLab:
  - Run a time-domain simulation
- The p-operator
  - Covered in Circuits I (this course)
- LaPlace transforms
  - Covered in Circuits II

Calculus I-III Method

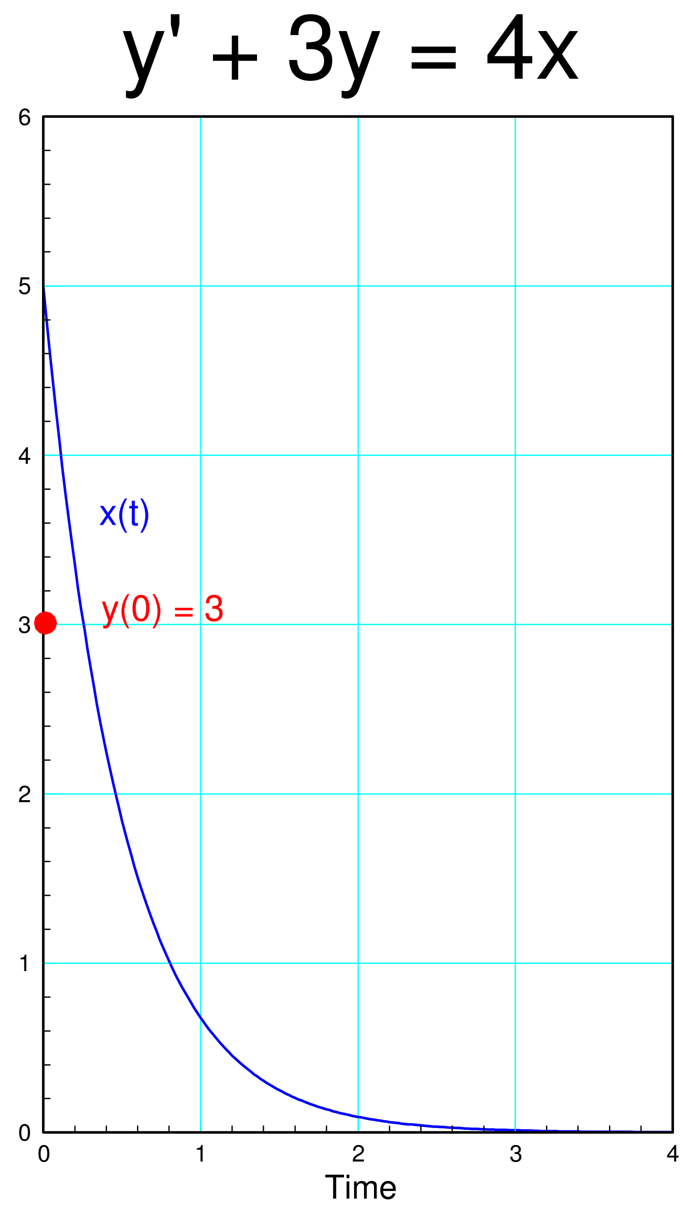
Assume you have a differential equation with a forcing function (x) and an in initial condition:

$$\frac{dy}{dt} + 3y = 4x$$

$$x(t) = 5e^{-2t}$$

$$y(0) = 3$$

Find y(t) for t>0



Forced Response

Using superposition, y(t) will contain two terms:

- y(t) assuming that x(t) = 0 (the natural response), and
- y(t) assuming that x(t) was on for all time (the forced response)

Start with the forced response

$$\frac{dy}{dt} + 3y = 4(5e^{-2t})$$

In calculus, you would guess the form of the solution

$$y(t) = ae^{-2t}$$

substitute for y(t) and dy/dt

$$(-2ae^{-2t}) + 3(ae^{-2t}) = 20e^{-2t}$$

Solve for 'a'

$$ae^{-2t} = 20e^{-2t}$$

$$a = 20$$

Natural Response

Once you get the forced response, find the zero-input solution

- Also known as the natural response

$$\frac{dy}{dt} + 3y = 0$$

In Calculus, you guess the form of the solution

$$y(t) = be^{-3t}$$

some lucky guessing involved here

Total Response

The total response is the forced response plus the natural response

$$y(t) = 20e^{-2t} + be^{-3t}$$

Plug in the initial condition

$$y(0) = 3 = 20 + b$$

Solve for b

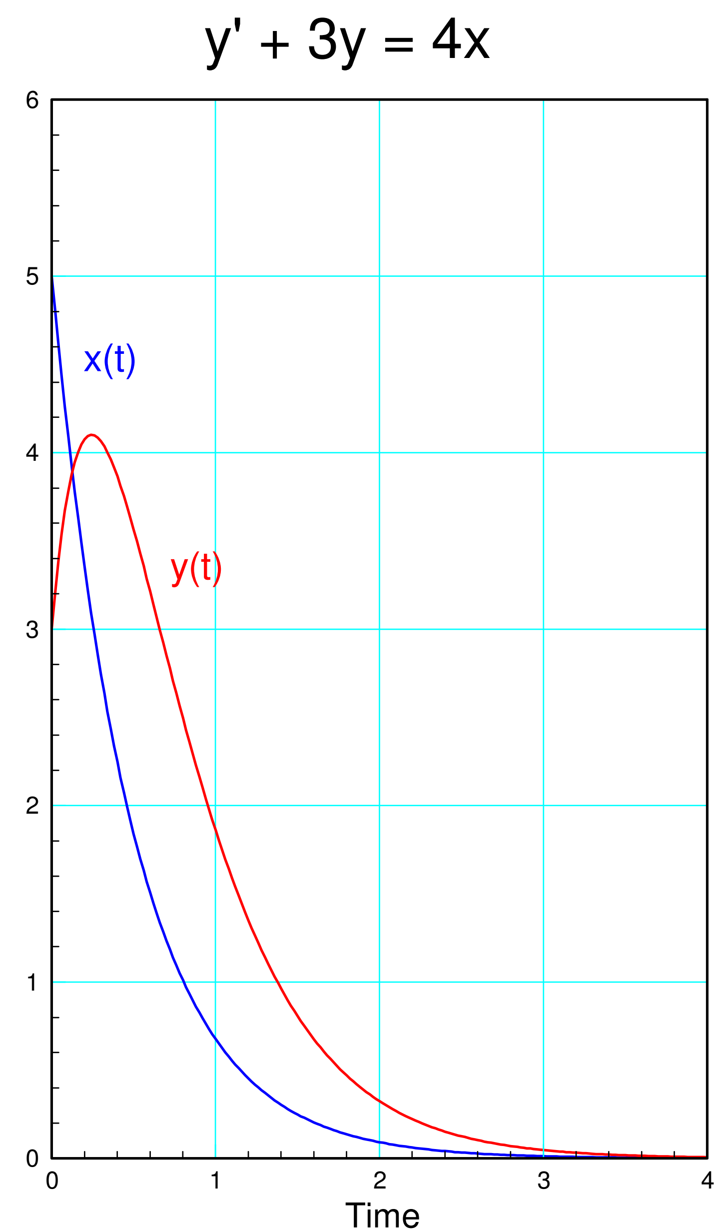
$$b = -17$$

giving you the total response for t>0:

$$y(t) = 20e^{-2t} - 17e^{-3t}$$

This is much too painful

- We need a better tool



The p-Operator

The p-operator is a trick that turns differential equations into algebraic equations in p

- The assumption is that algebra is easier than calculus

The basic assumption behind the p-operator is that all functions are in the form of

$$x(t) = e^{pt}$$

With this assumption, differentiation is equivalent to multiplying by p

$$\frac{dx}{dt} = pe^{pt} = px$$

Differential Equations and Transfer Functions

Using the p-operator, the differential equation

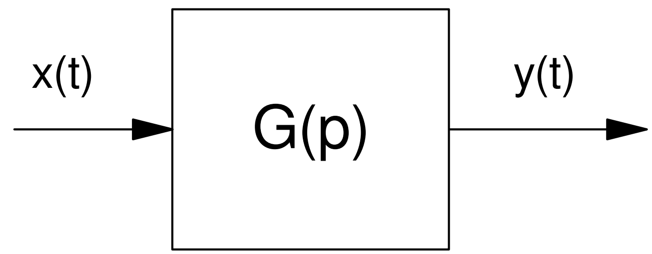
$$\frac{dy}{dt} + 3y = 4x$$

becomes

$$pY + 3Y = 4X$$

or

$$Y = \left(\frac{4}{p+3}\right) X = G(p) \cdot X$$



The term $G(p)$ is termed *the transfer function*.

- This behaves like a gain from X to Y
- Note that the gain depends upon p

Forced Response with Different Inputs:

Assume X and Y are related by the following differential equation

$$\frac{dy}{dt} + 3y = 4x$$

Find the forced response assuming

- $x(t) = 5$
- $x(t) = 5e^{-2t}$
- $x(t) = 5\sin(2t)$

Case 1: Constant Inputs

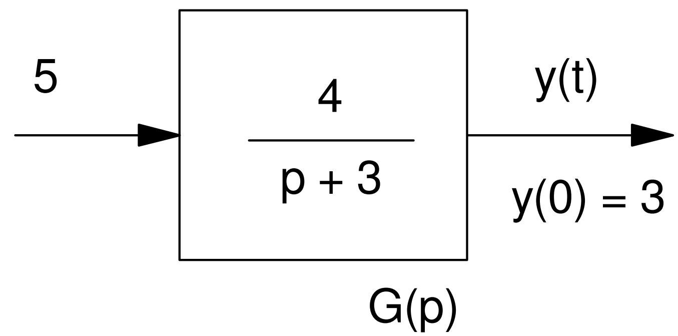
This is really a phasor problem. X and Y are related by

$$Y = \left(\frac{4}{p+3}\right)X$$

Assume

$$x(t) = 5$$

$$y(0) = 3$$



The assumption behind the p-operator is that all functions are in the form of

$$x(t) = ae^{pt}$$

Matching terms, a=5 and p=0

$$5 = a \cdot e^{pt}$$

$$5 = 5 \cdot e^{0t}$$

Natural Response

Now add in the natural response

- The response when x(t) = 0

The transfer funcion for X=0 is:

$$Y = \left(\frac{4}{p+3}\right) \cdot 0$$

$$(p+3)Y = 0$$

This means that either

- Y = 0 *the trivial solution*
- p = -3

Going with the latter, y(t) is of the form

$$y(t) = be^{-3t}$$

Total Response

The total response is

- The forced response plus
- The natural response

$$y(t) = \frac{20}{3} + be^{-3t}$$

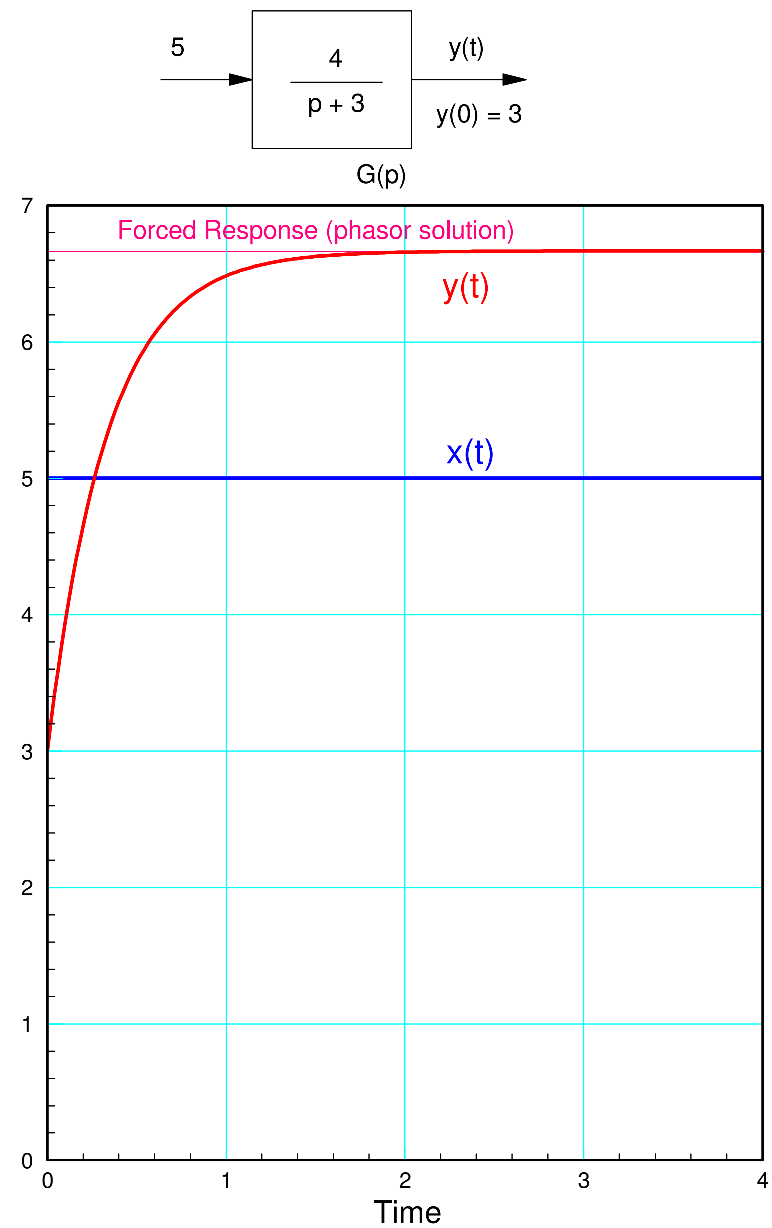
Plugging in the initial conditions

$$y(0) = 3 = \frac{20}{3} + b$$

$$b = -\frac{11}{3}$$

meaning

$$y(t) = \frac{20}{3} - \frac{11}{3}e^{-3t}$$



Case 2: Exponential Input

Next, consider the case where x(t) is a decaying exponential

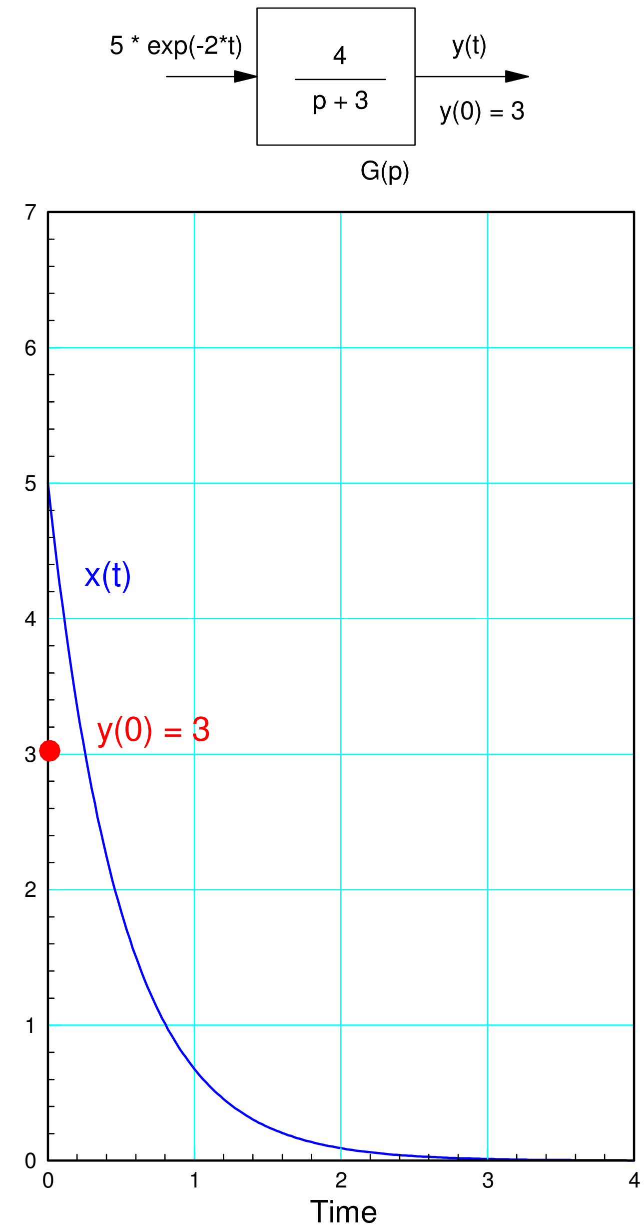
$$x(t) = 5e^{-2t}$$

Again, the basic assumption behing the p-operator is that all functions are in the form of

$$x(t) = a \cdot e^{pt}$$

Meaning

- a = 5
- p = -2



Case 2: Exponential Input

- Continued

Going back to the transfer function

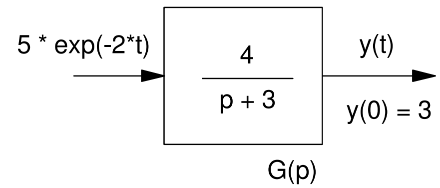
$$Y = \left(\frac{4}{p+3}\right) X$$

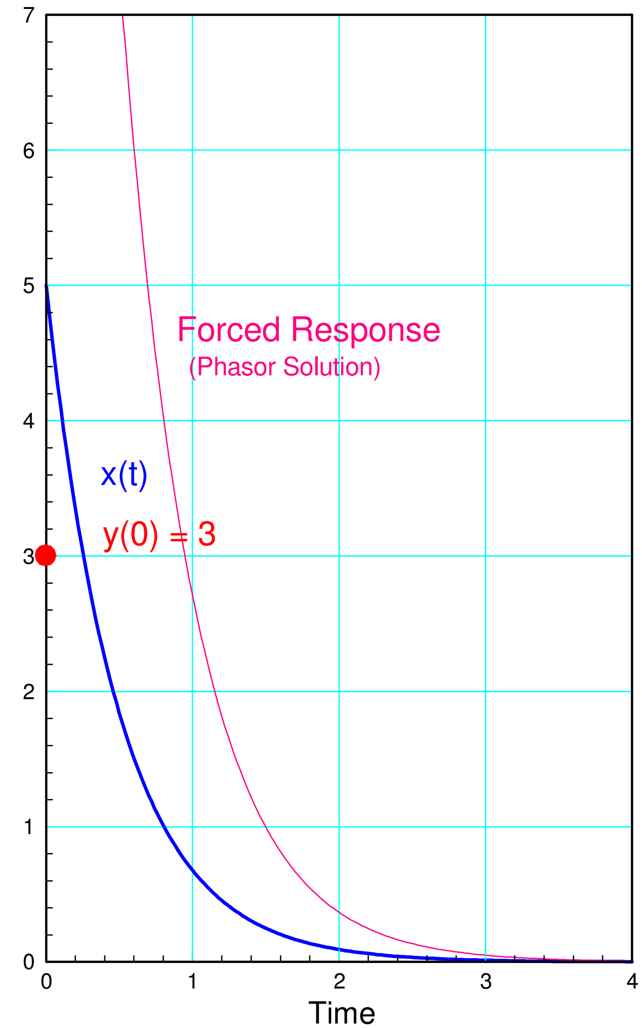
$$Y = \left(\frac{4}{p+3}\right)_{p=-2} \cdot (5)$$

$$Y = 20$$

meaning

$$y(t) = 20e^{-2t}$$





Case 2: Total Response

The total response is the sum of

- The forced response plus
- The natural response

$$y(t) = 20e^{-2t} + be^{-3t}$$

Plug in the initial condition

$$y(0) = 3 = 20 + b$$

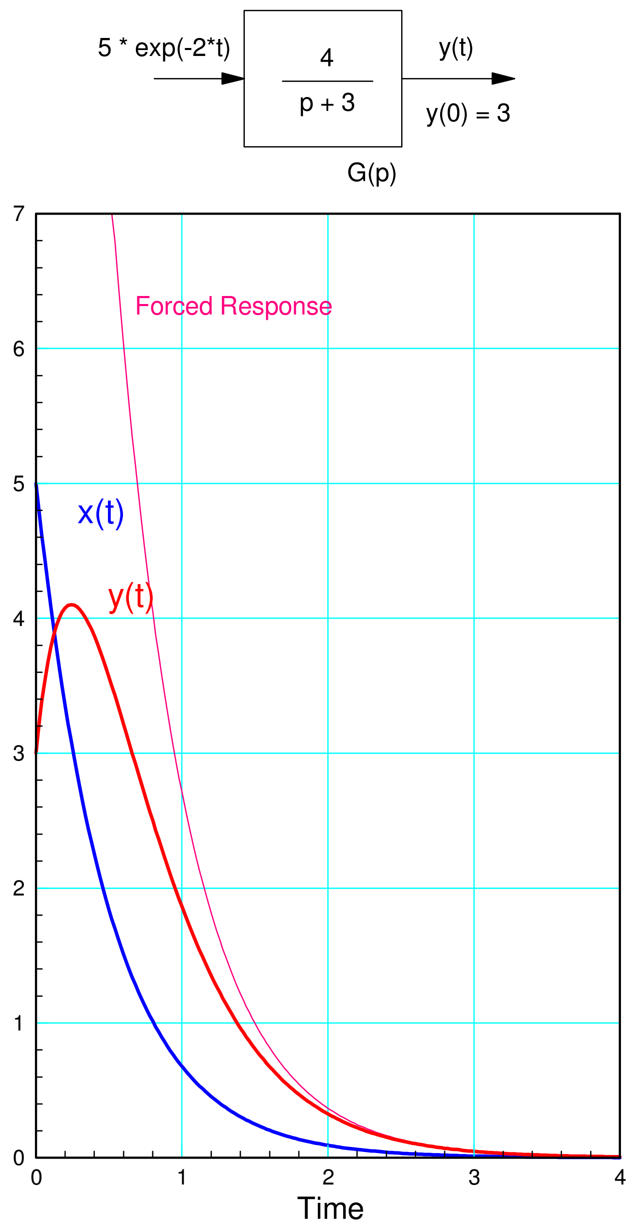
$$b = -17$$

giving

$$y(t) = 20e^{-2t} - 17e^{-3t} \qquad \text{t} > 0$$

Note that this is the same result we got using calculus

- Only a *lot* easier



Case 3: Sinusoidal Input

Finally, consider the case where x(t) is a sinusoid

$$x(t) = 5\sin(2t)$$

$$y(0) = 3$$

From phasors, this is

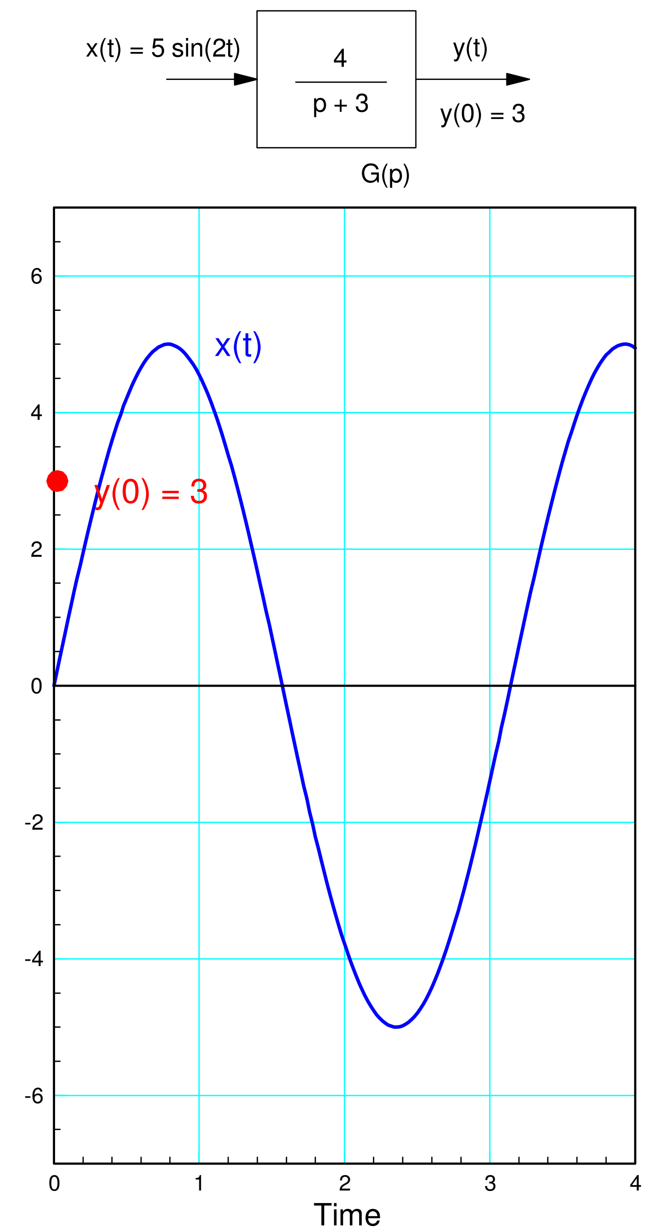
$$X = 0 - j5$$

$$5\sin(2t) = real((0 - j5)e^{j2t})$$

meaning

$$p = j2$$

$$X = 0 - j5$$



Case 3: Sinusoidal Input

- continued

Y is then

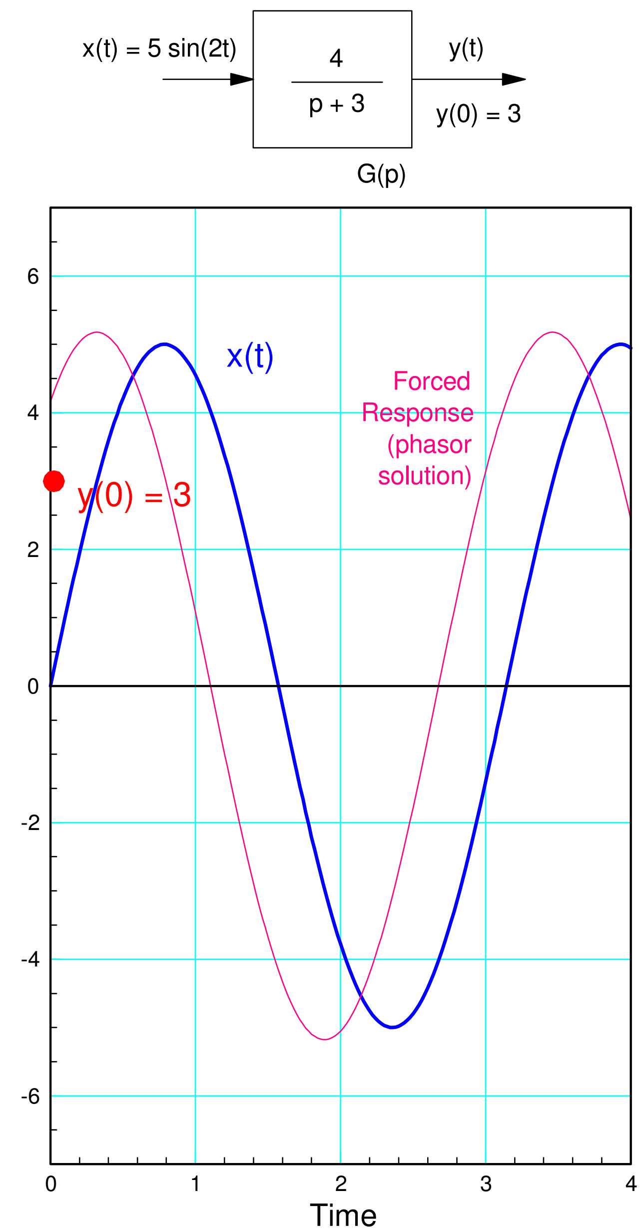
$$Y = \left(\frac{4}{p+3}\right) X$$

$$Y = \left(\frac{4}{p+3}\right)_{p=j2} \cdot (0 - j5)$$

$$Y = 4.1654 - j3.0769$$

This is the phasor representation of

$$y(t) = 4.1654\cos(2t) + 3.0769\sin(2t)$$



Case 3: Sinudoidal Input (cont'd)

The total response is then

$$y(t) = 4.1654\cos(2t) + 3.0769\sin(2t) + be^{-3t}$$

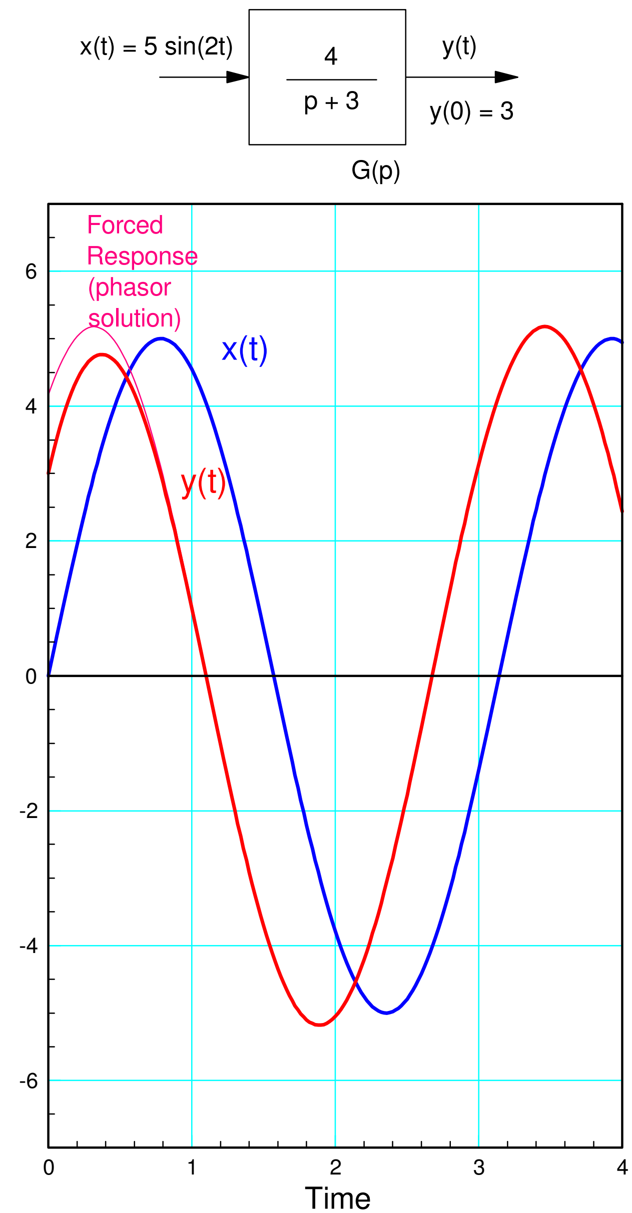
Plugging in the initial condition

$$y(0) = 3 = 4.1654 + b$$

$$b = -1.1654$$

y(t) is then

$$y(t) = 4.1654\cos(2t) + 3.0769\sin(2t)$$
$$-1.1654e^{-3t} \qquad t>0$$



2nd-order Differential Eq

Finally, use the p-operator to solve

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 5y = 10x$$

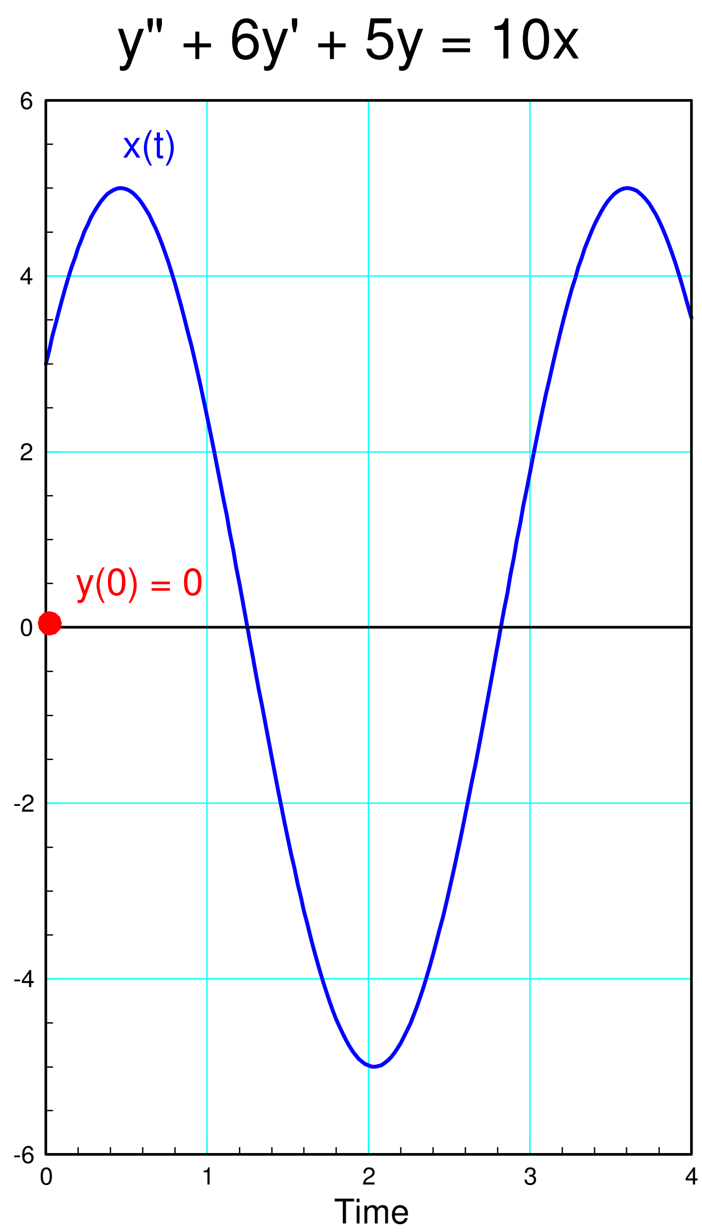
where

$$y(0) = 0$$

$$\frac{dy}{dt}(0) = 0$$

and

$$x(t) = 3\cos(2t) + 4\sin(2t)$$



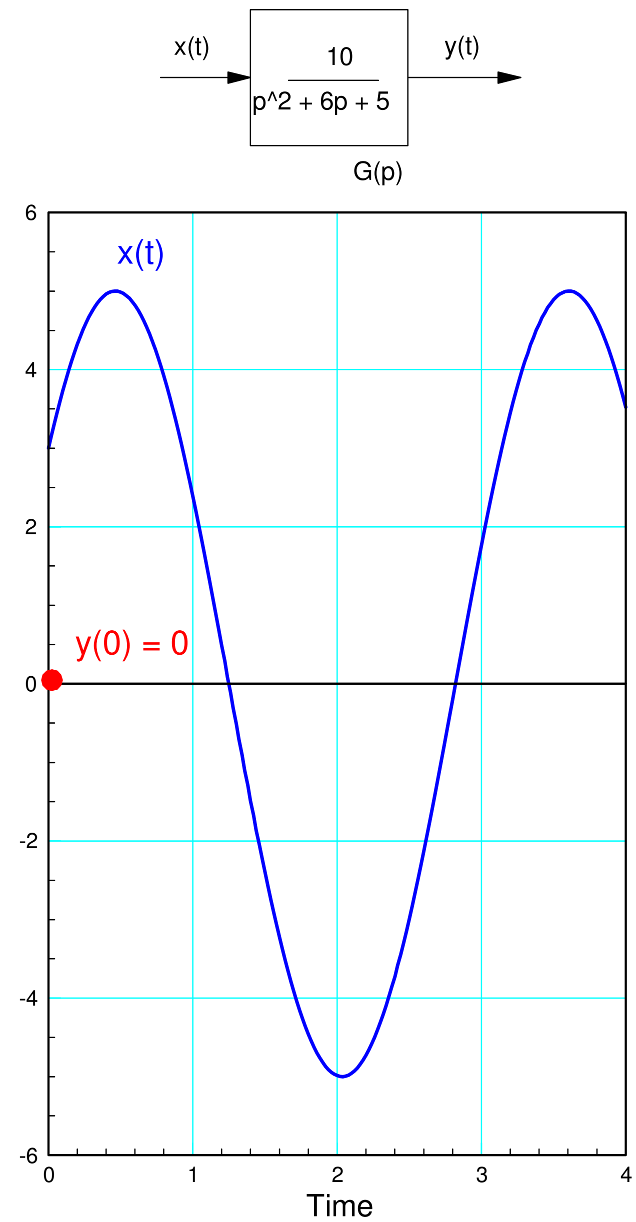
2nd-order Differential Eq

Start with the p-operator

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 5y = 10x$$

$$p^2 Y + 6pY + 5Y = 10X$$

$$Y = \left(\frac{10}{p^2+6p+5}\right) X$$



Use phasor analysis to find the forced response

$$Y = \left(\frac{10}{p^2+6p+5}\right) X$$

$$x(t) = 3\cos(2t) + 4\sin(2t)$$

In phasor form

$$p = j2$$

$$X = 3 - j4$$

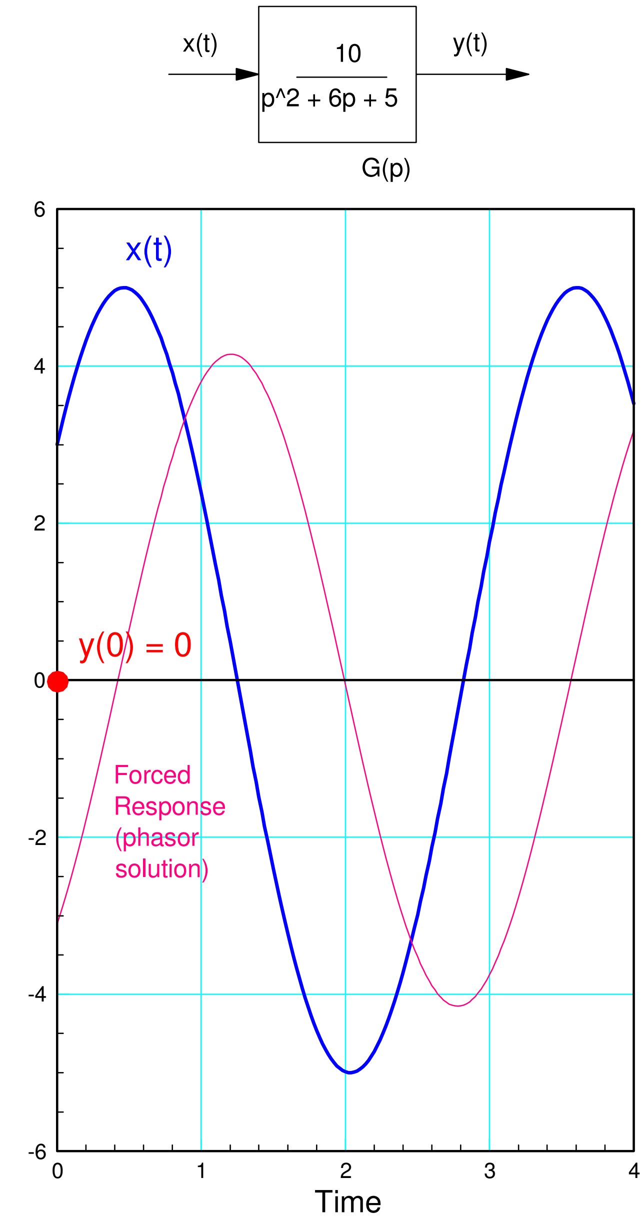
Y is then

$$Y = \left(\frac{10}{p^2+6p+5}\right)_{p=j2} \cdot (3 - j4)$$

$$Y = -3.1034 - j2.7586$$

meaning

$$y(t) = -3.1034\cos(2t) + 2.7586\sin(2t)$$



Now find the natural response

$$Y = \left(\frac{10}{p^2+6p+5}\right) \cdot 0$$

$$(p^2 + 6p + 5)Y = 0$$

$$(p+1)(p+5)Y = 0$$

Either

- Y = 0 *trivial solution, or*
- p = (-1, -5)

Going with the latter

$$y(t) = ae^{-t} + be^{-5t}$$

The total response is

y(t) = forced response + natural response

$$y(t) = (-3.1034\cos(2t) + 2.7586\sin(2t)) + (ae^{-t} + be^{-5t})$$

Plug in the initial conditions

$$y(0) = 0 = -3.1034 + a + b$$

$$y'(0) = 0 = 2 \cdot 2.7586 - a - 5b$$

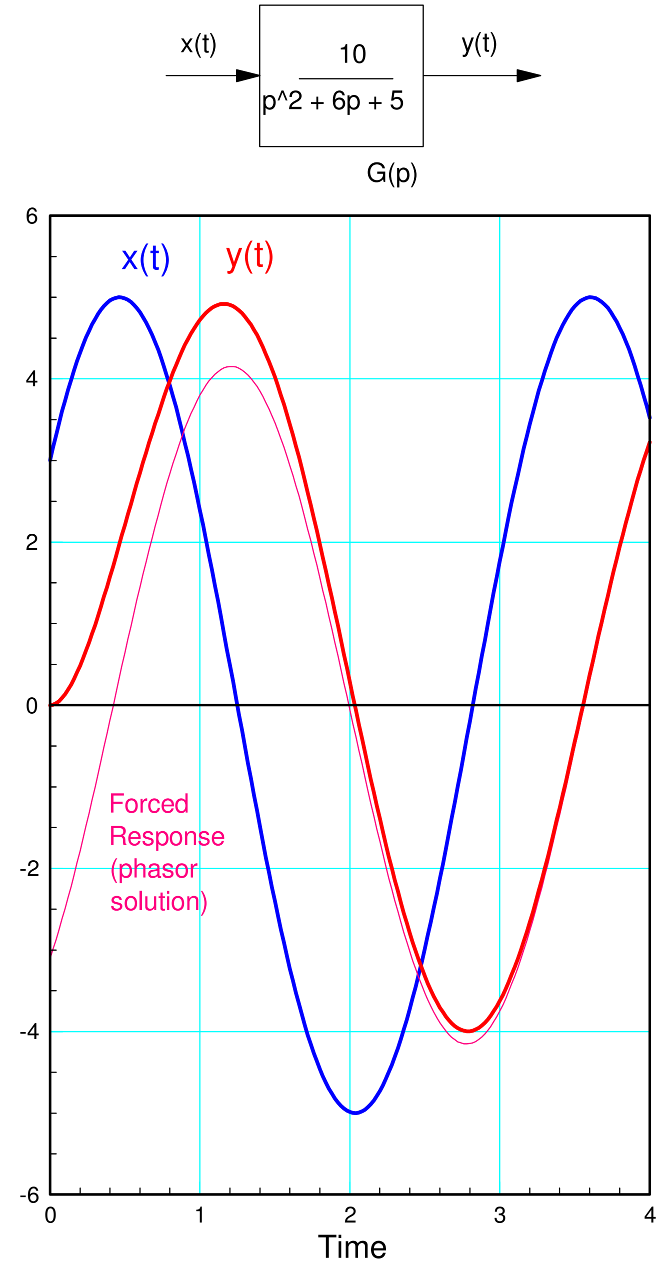
Solving (takes some algebra)

$$a = 2.5000$$

$$b = 0.6034$$

so

$$y(t) = (-3.1034\cos(2t) + 2.7586\sin(2t)) + (2.5e^{-t} + 0.6034e^{-5t})$$



Summary

The p-operator assumes all functions are of the form

$$y(t) = e^{pt}$$

The p-opeator makes solving differential equations a lot easier

- Replace each derivitive with a 'p'
- Use algebra to find the transfer function

This results in the forced response being a phasor problem

- Evaluate the gain at the frequency of the input
- Times the input at that frequency

The natural response is determined by the roots of the denominator polynomial