Fourier Transform

Solving differential equations when the input is periodic

ECE 211 - Lecture #32

Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

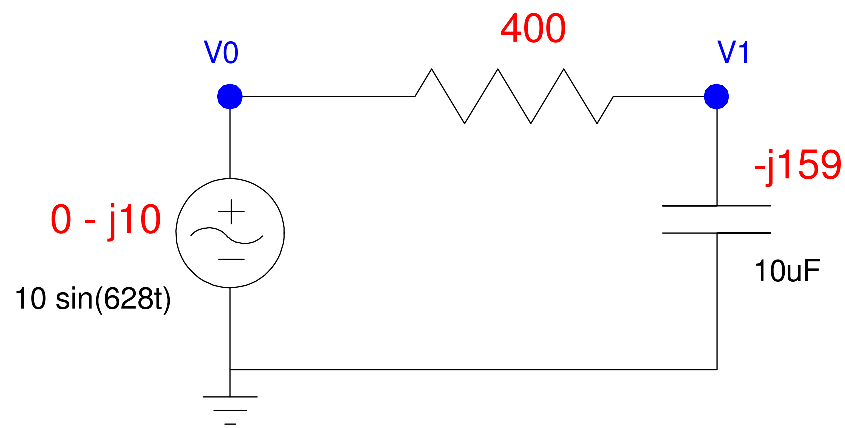
Recap: Phasors

If a circuit has a sinusoidal input, use phasor analysis

- Replace voltage & current sources with their phasor equivalent
- Replace inductors & capacitors with their phasor equivalent

Analyze as before

- Only now using complex numbers



Recap: Superposition

If a circuit has

- Two different inputs, or
- An input containging two or more frequencies

Use the property of linearity

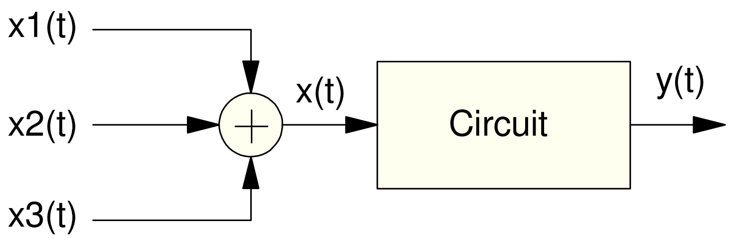
$$f(a+b) = f(a) + f(b)$$

Treat this as separate problems

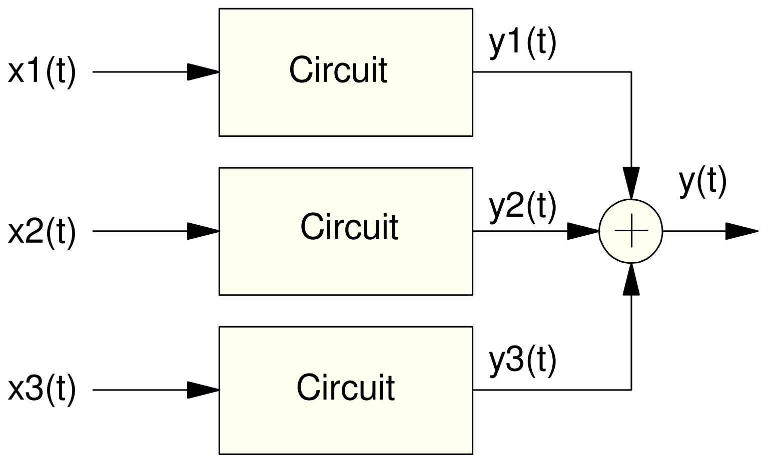
- Different problem for each input
- Different problem for each frequecy

Solve each problem separately

The total answer is the sum of the separate answers



Actual Problem

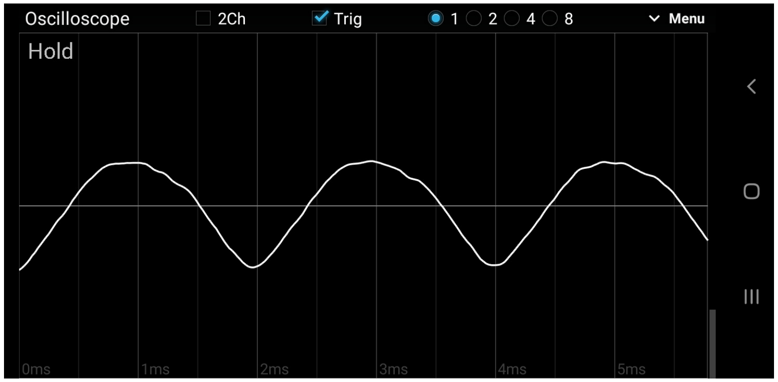


Analysis Using Superposition

Problem:

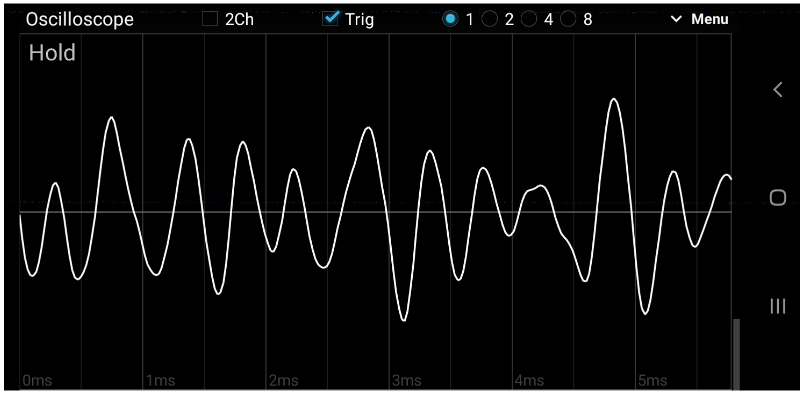
How do you solve for voltages and currents when

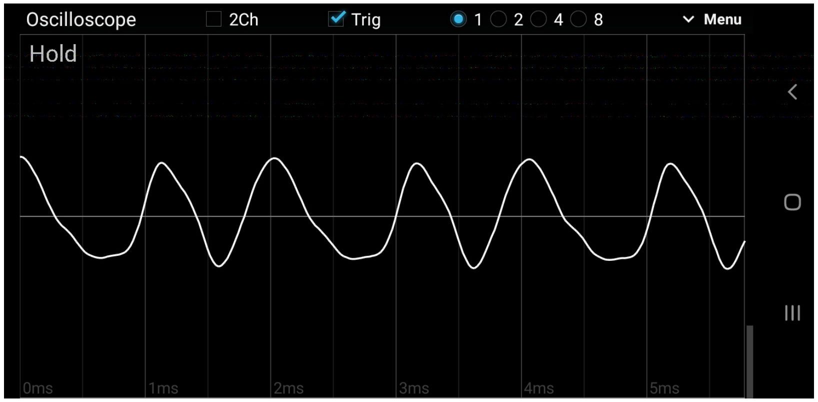
- The input is periodic,
- But *not* a sinusoid?



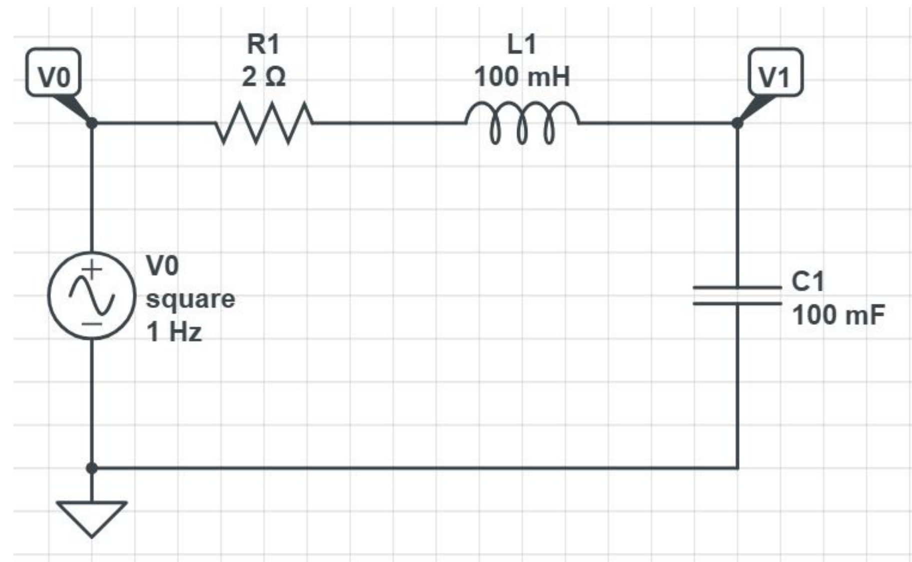
Example Note A4 (500Hz):

- Piano
- Tubular Bells
- Cello





CircuitLab Example



RLC Circuit

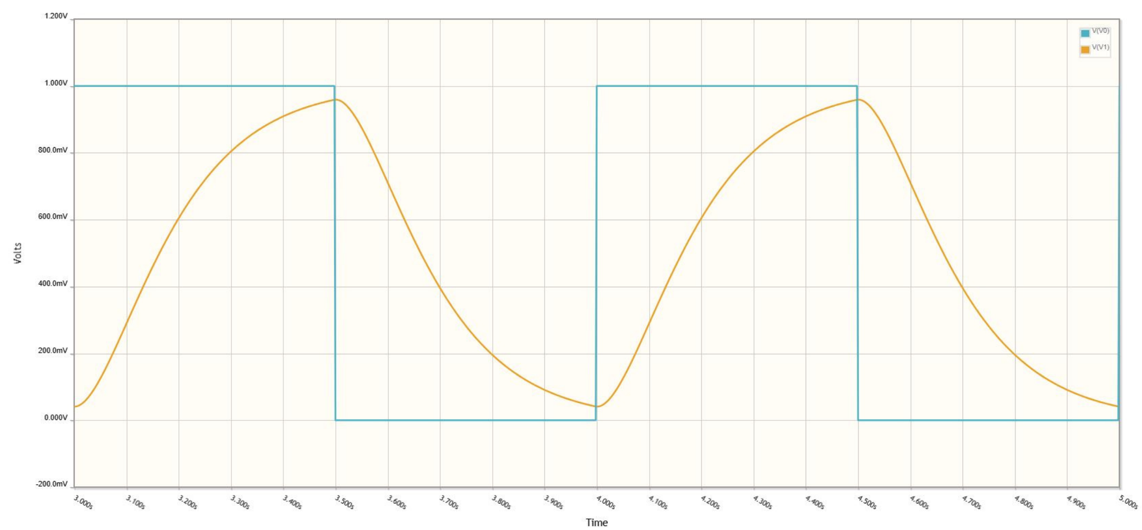
- By voltage division
- $V_2 = \left(\dfrac{\frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}} \right) V_1$
- $V_2 = \left(\dfrac{1}{(j\omega)^2 LC + (j\omega)RC + 1} \right) V_1$

Set V0 to be a square wave input

- 1Hz
- 0.5V offset
- 0.5V amplitude

V1(t) found using CircuitLab

- How do you compute V1(t)
- When V0(t) is a square wave?



Solution: Fourier Transforms

Change the problem

- Express the signal in terms of sine waves

Note:

- Fourier Transforms will be covered in more detail in Signals & Systems

However, once you have

- Phasors, and
- Superposition

you have Fourier transforms.

- We're ready for it.

Trick Used in Fourier Transforms

Take a signal that is periodic in time T

$$x(t) = x(t+T)$$

Approximate it as the sum of sine waves

- Each is also periodic in time T

$$x(t) = \sum a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

$$\omega_0 = \frac{2\pi}{T}$$

Now that x(t) is in terms of sine waves, use phasor analysis at each frequency

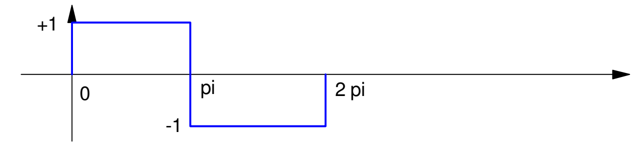
Note:

- Equals is a powerful symbol
- It means the two sides are equivalent
  - Any time you see x(t), you can replace it with the Fourier series
  - Any time you see the Fourier series, you can replace it with x(t)

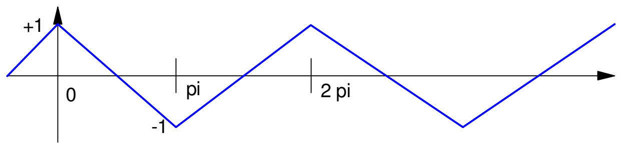
Problem: How to go right to left

- If you know the formula for x(t) in terms of sine waves
- Simply add them together in Matlab to get x(t) in terms of time

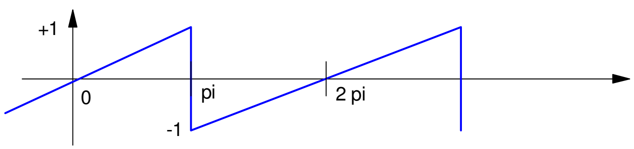
Table of Fourier Transforms (CRC Handbook of Mathematics)



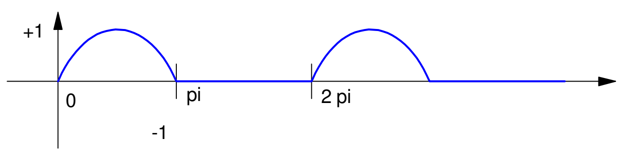
$$x(t) = \frac{4}{\pi} \sum_{\text{n odd}} \left(\frac{1}{n}\right) \sin(nt)$$



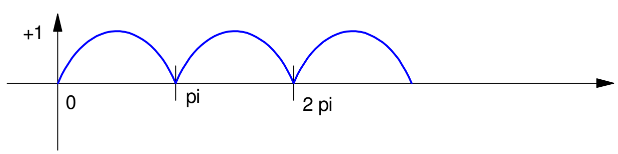
$$x(t) = \frac{8}{\pi^2} \sum_{\text{n odd}} \left(\frac{1}{n^2}\right) \cos(nt)$$



$$x(t) = \frac{2}{\pi} \sum_{n} (-1)^{n-1} \left(\frac{1}{n}\right) \sin(nt)$$



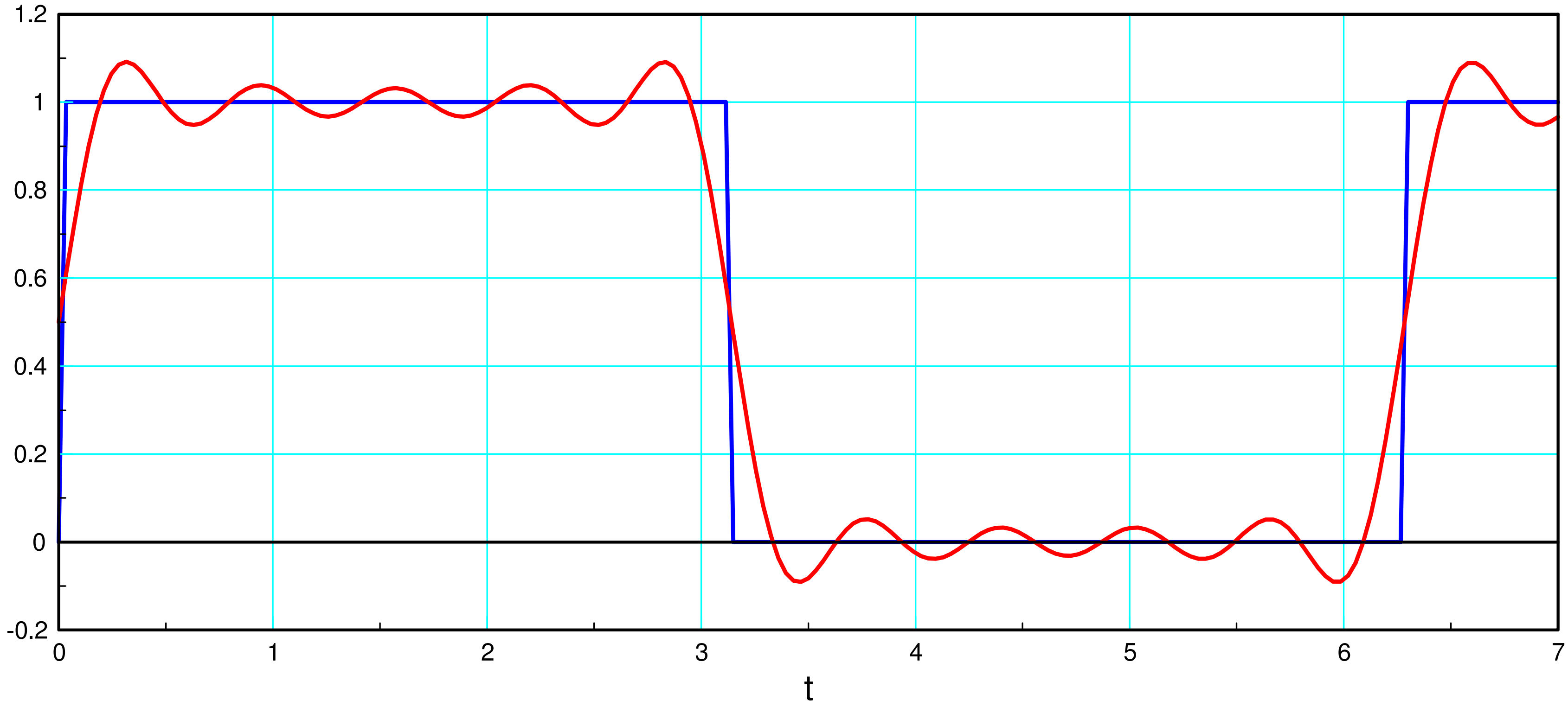
$$x(t) = \frac{1}{\pi} + \frac{1}{2}\sin(x) - \frac{2}{\pi} \sum_{\text{n even}} \frac{1}{n^2-1} \cos(nt)$$



$$x(t) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{\text{n even}} \frac{1}{n^2-1} \cos(nt)$$

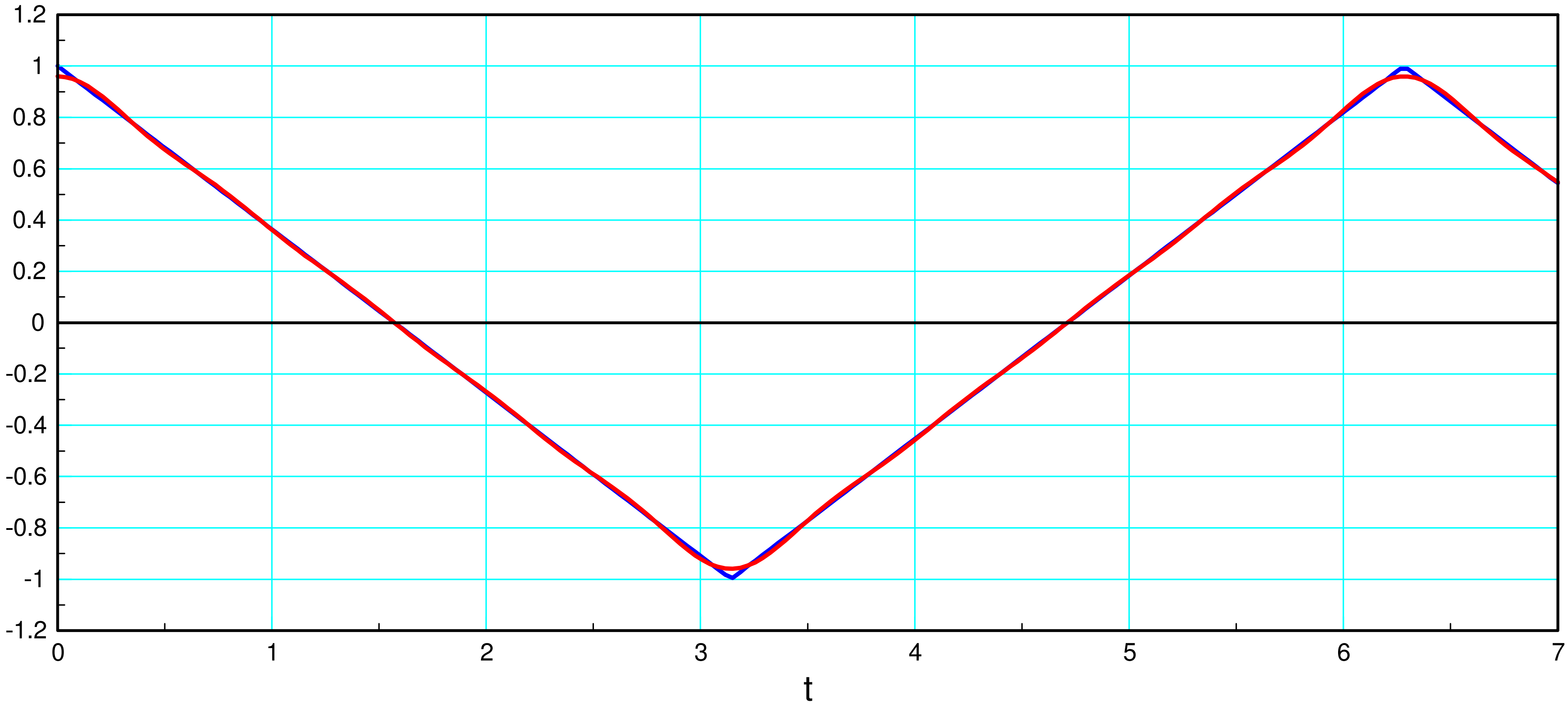
Example: Square Wave

- $x(t) = 0.5 + \sum_{\text{n odd}} \frac{2}{\pi n} \sin(nt)$
- n = 9 (red) & infinity (blue)



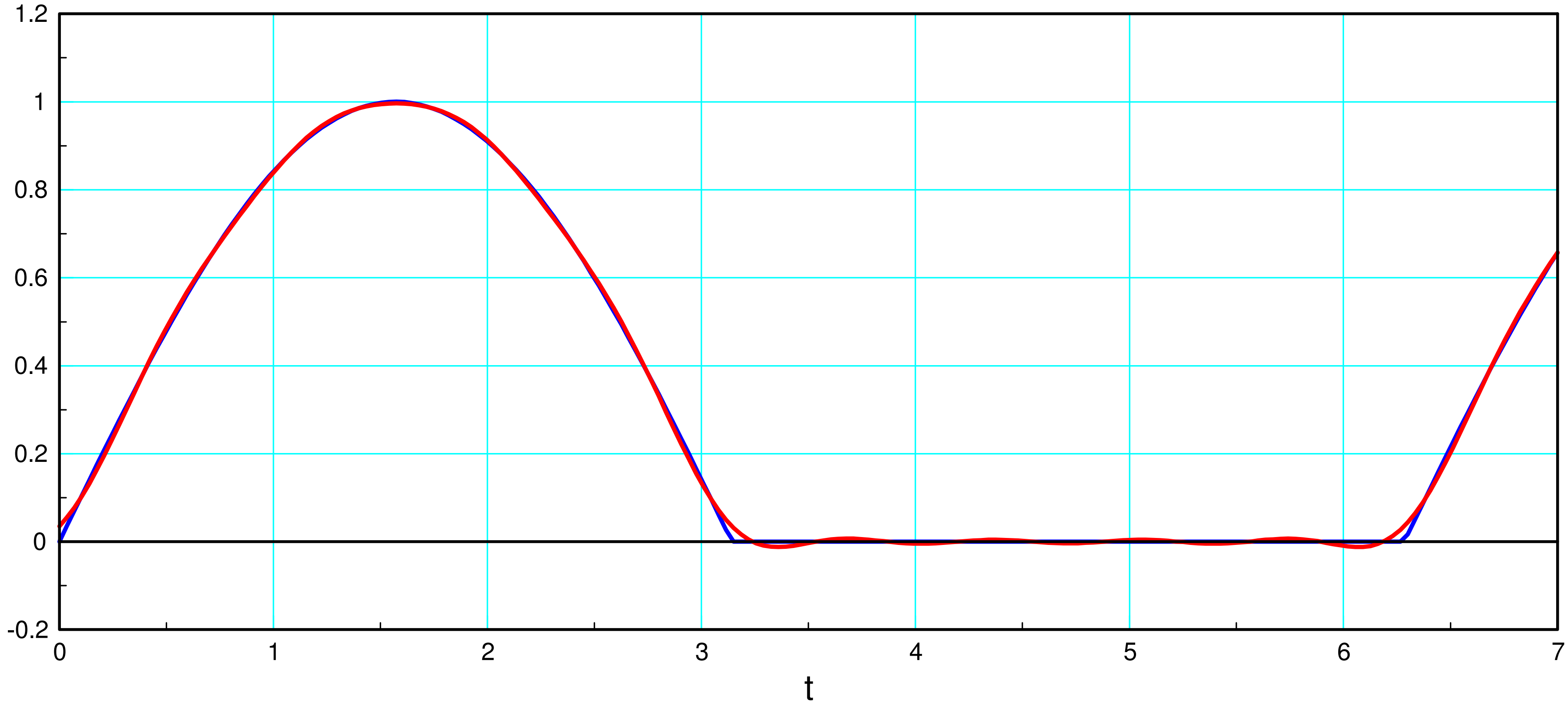
Example: Triangle Wave

- $x(t) = \frac{8}{\pi^2} \sum_{\text{n odd}}^{\infty} \frac{1}{n^2} \cos(nt)$
- n = 9 (red) & infinity (blue)



Example: 1/2 Wave Rectified Sine Wave

- $x(t) = \frac{1}{\pi} + \frac{1}{2}\sin(x) - \frac{2}{\pi}\sum_{\text{n even}} \frac{1}{n^2-1}\cos(nt)$
- n = 9 (red) & infinity (blue)



Problem: How do you go left to right?

- Given a periodic x(t), how do you decompose it into sine waves?

$$x(t) = a_0 + a_1\cos(\omega_0 t) + b_1\sin(\omega_0 t) + a_2\cos(2\omega_0 t) + b_2\sin(2\omega_0 t) + \ldots$$

$$a_i = ? \qquad b_i = ?$$

$$x(t) \xrightarrow{\;?\;} \text{frequency content}$$

Method #1: Curve Fitting

You can curve fit any funciton.

Example: Assume x(t) is a square wave

$$x(t) = x(t+T)$$

$$x(t) = \begin{cases} +1 & 0 < t < \frac{T}{2} \\ 0 & \frac{T}{2} < t < T \end{cases}$$

$$T = 1$$

Curve fit x(t) to a power series

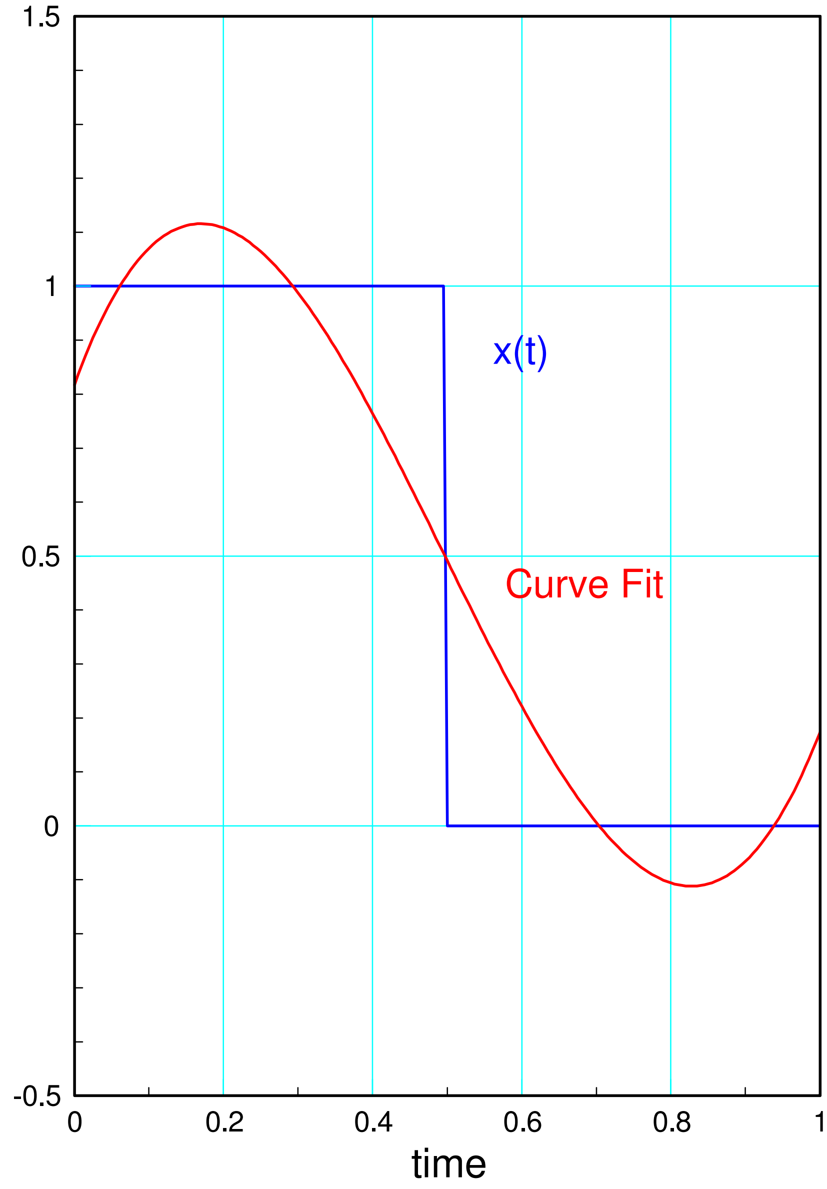
$$x(t) \approx a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4$$

Power Series (cont'd)

In Matlab

```
>> T = 1;
>> t = [0:0.001:T]';
>> x = 1*(t<T/2);
>> B = [t.^0, t, t.^2, t.^3, t.^4];
>> A = inv(B'*B)*B'*x

a0     0.8134
a1     3.7592
a2   -13.2293
a3     8.9589
a4    -0.1175

>> plot(t,x,'b',t,B*A,'r')
```

The curve fit gets better if you add more terms



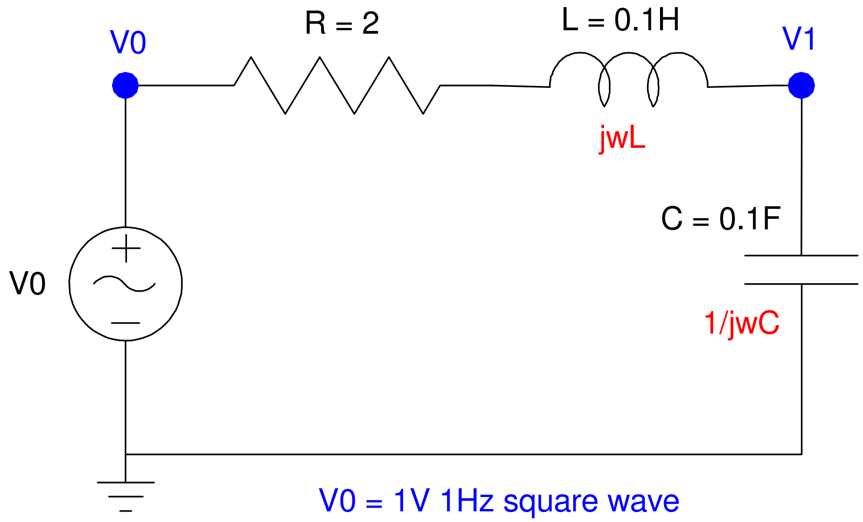
Problem with a Power Series

x(t) is expressed in terms of simpler functions

- good

The basis elements aren't useful for circuit analysis

- they're not sine waves
- phasor analysis doesn't work
- bad



Curve fit x(t) to a sine-waves

Repeat: Make the period 2π ($\omega_0 = 1$)

- Change of variable: doesn't affect coefficients

Express x(t) in terms of harmonics

$$x(t) = a_0 + a_1\cos(t) + b_1\sin(t) + a_2\cos(2t) + b_2\sin(2t) + a_3\cos(3t) + b_3\sin(3t) + \ldots$$

In Matlab

```
>> T = 2*pi;
>> t = [0:0.001:T]';
>> x = 1*(t<T/2);
>> B = [t.^0, cos(t), sin(t), cos(2*t), sin(2*t), cos(3*t), sin(3*t)];
>> A = inv(B'*B)*B'*x

a0     0.5000
a1     0.0001
b1     0.6366
a2    -0.0000
b2    -0.0000
a3     0.0001
b3     0.2122
```

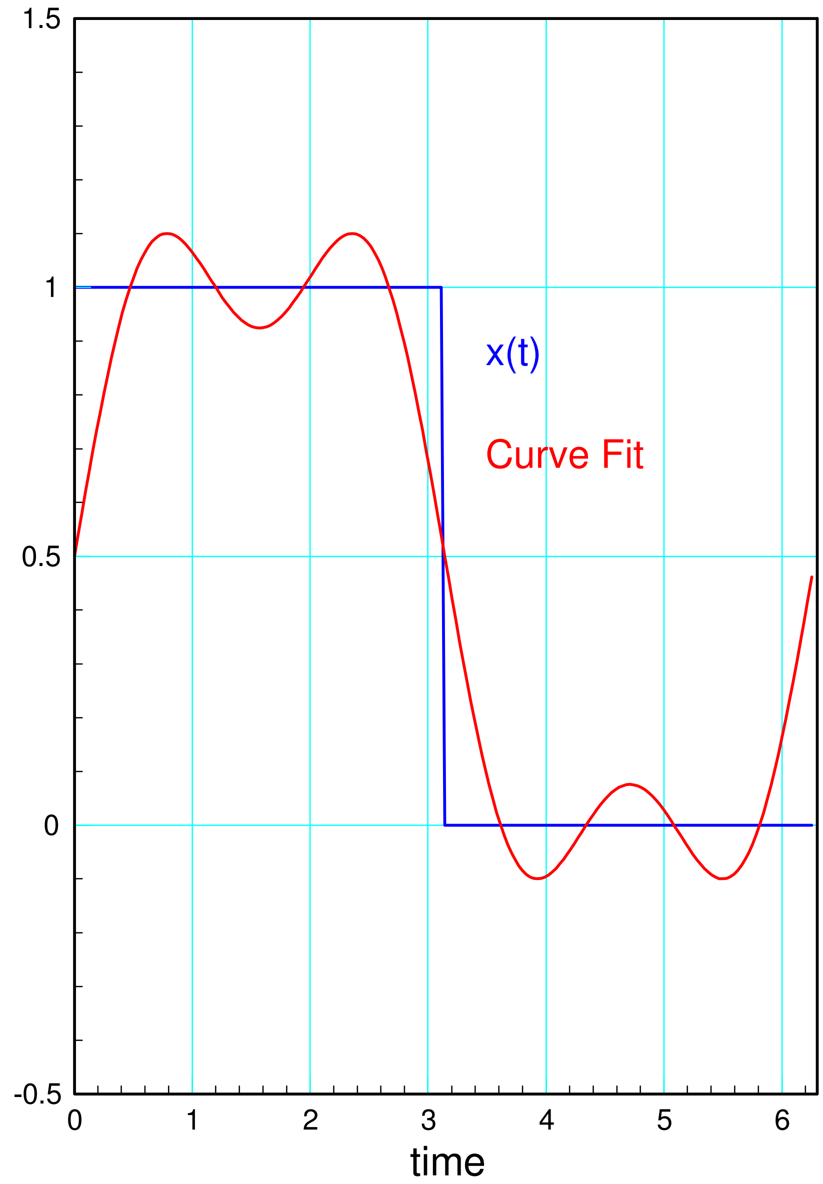
Comment

Result:

$$x(t) = 0.5 + 0.6366\sin(t) + 0.2122\sin(3t)$$

This is just a curve fit

- But, it's a useful curve fit
- x(t) is expressed in terms of sine waves
- Phasor analysis & superposition can now be used



Method #2: Fourier Transforms

Note that the basis used is an orthogonal basis

$$\int_T \cos(n\omega_0 t) \cdot \sin(m\omega_0 t) \cdot dt = 0$$

$$\int_T \cos(n\omega_0 t) \cdot \cos(m\omega_0 t) \cdot dt = 0 \qquad m \neq n$$

$$\int_T \sin(n\omega_0 t) \cdot \sin(m\omega_0 t) \cdot dt = 0 \qquad m \neq n$$

It is *not* an ortho-normal basis though

$$\int_T 1 \cdot 1 = T$$

$$\int_T \cos(n\omega_0 t) \cdot \cos(n\omega_0 t) dt = \frac{T}{2}$$

$$\int_T \sin(n\omega_0 t) \cdot \sin(m\omega_0 t) dt = \frac{T}{2}$$

This allows you to find each term separately

Fourier Transforms (cont'd)

Recall

$$x(t) = a_0 + a_1 \cos(t) + b_1 \sin(t) + a_2 \cos(2t) + b_2 \sin(2t) + a_3 \cos(3t) + b_3 \sin(3t) + \ldots$$

Each term can be found using

$$a_0 = \frac{1}{T} \int_T x(t) \cdot 1 \cdot dt$$

$$a_n = \frac{1}{2T} \int_T x(t) \cdot \cos(nt) \cdot dt$$

$$b_n = \frac{1}{2T} \int_T x(t) \cdot \sin(nt) \cdot dt$$

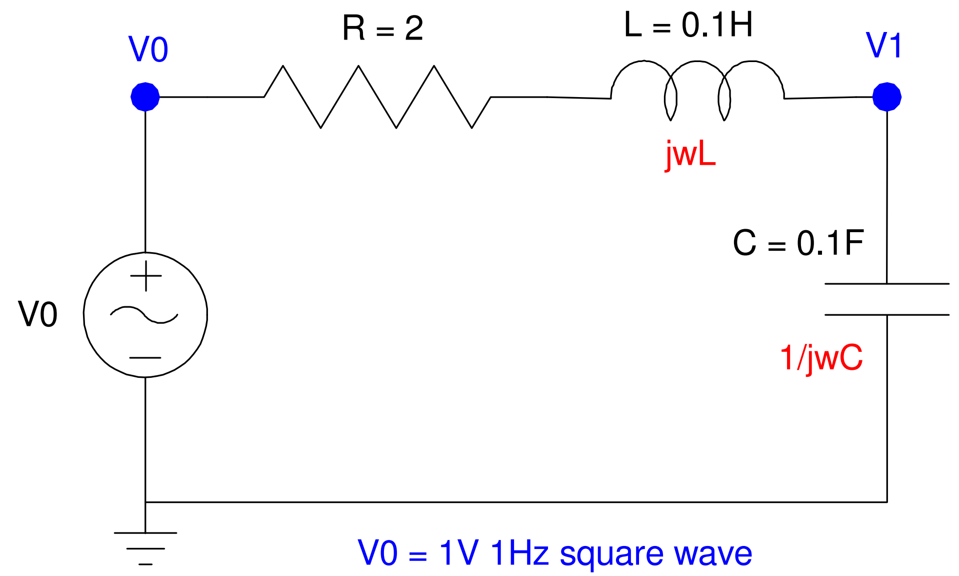
In Matlab

```
>> T = 2*pi;
>> t = [0:0.001:T]';
>> x = 1*(t<T/2);
>> a0 = mean(x)
a0 =     0.5000

>> a1 = 2 * mean(x .* cos(t))
a1 =  1.8862e-004

>> b1 = 2 * mean(x .* sin(t))
b1 =     0.6365

>> a2 = 2 * mean(x .* cos(2*t))
a2 =  1.2965e-004

>> b2 = 2 * mean(x .* sin(2*t))
b2 = -7.6835e-008

>> a3 = 2 * mean(x .* cos(3*t))
a3 =  1.8862e-004

>> b3 = 2 * mean(x .* sin(3*t))
b3 =     0.2122
```

Circuit Analysis

Going back to our original circuit, find V1(t)

$$V_1 = \left(\frac{1}{LC(j\omega)^2 + RC(j\omega) + 1}\right) V_0$$



Express V0(t) in term sof sine waves

- use x(t) from previous analysis
- change the period from 2π
- to 1Hz

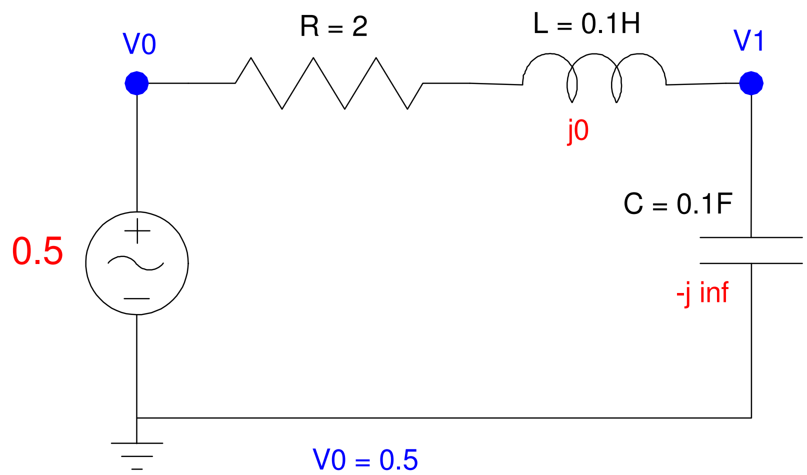
Result:

$$\begin{aligned} V_0(t) = {} & 0.5 \\ & +0.6365 \sin(2\pi t) \\ & +0.2122 \sin(6\pi t) \end{aligned}$$

DC Analysis

Use superposition and phasors

- V0(t) contains three terms
- V2(t) contains three terms

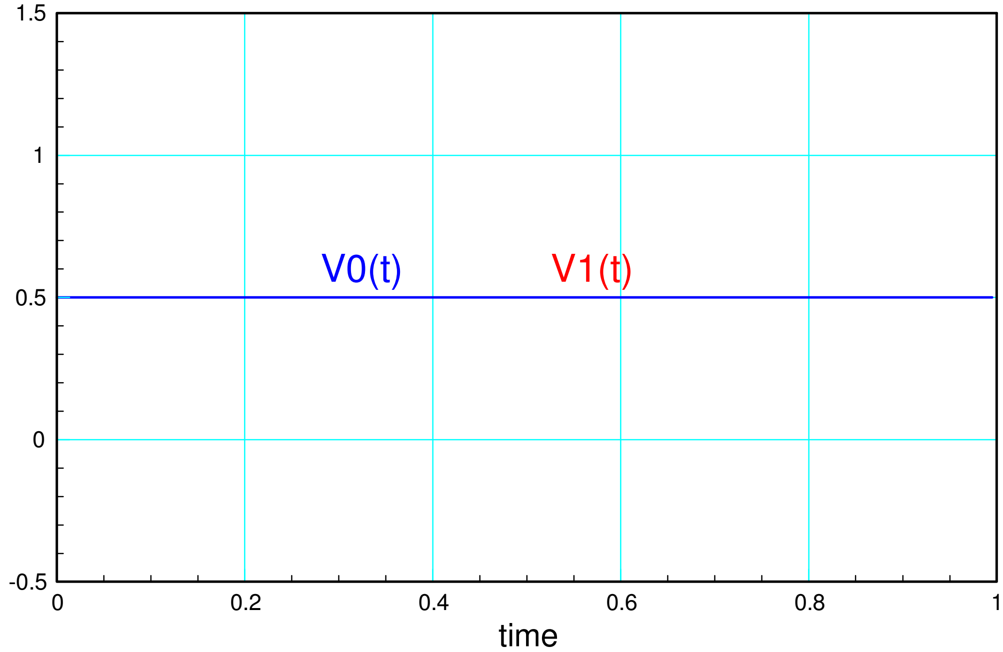


DC Analysis

$$V_0(t) = 0.5$$

$$V_1 = \left(\frac{1}{LC(j\omega)^2 + RC(j\omega) + 1} \right)_{\omega=0} V_0$$

$$V_1 = 0.5$$



1Hz Analysis

$$v_0(t) = 0.6365\sin(2\pi t)$$

In phasor form

$$V_0 = 0 - j0.6365$$

$$\omega = 2\pi$$

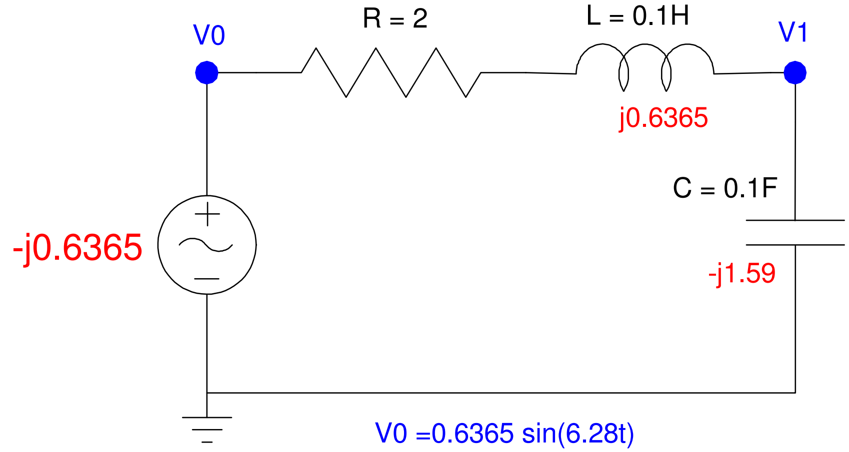
V1 is then

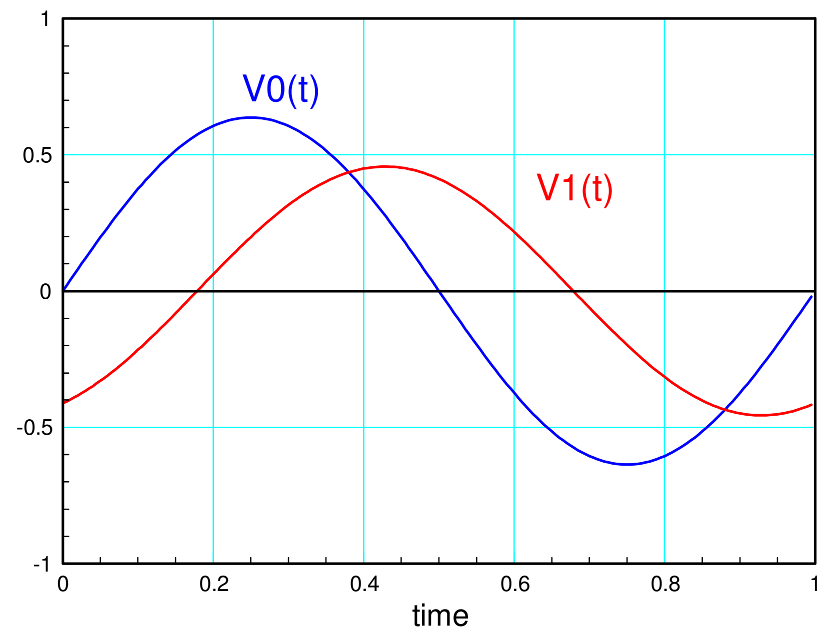
$$V_1 = \left(\frac{2}{(j0.04\omega)^2 + (j0.04\omega) + 1}\right)_{\omega=2\pi}(-j0.6365)$$

$$V_1 = \text{-}0.3397 \text{ - } 1.2678\text{i}$$

meaning

$$v_1(t) = -0.340\cos(2\pi t) + 1.268\sin(2\pi t)$$



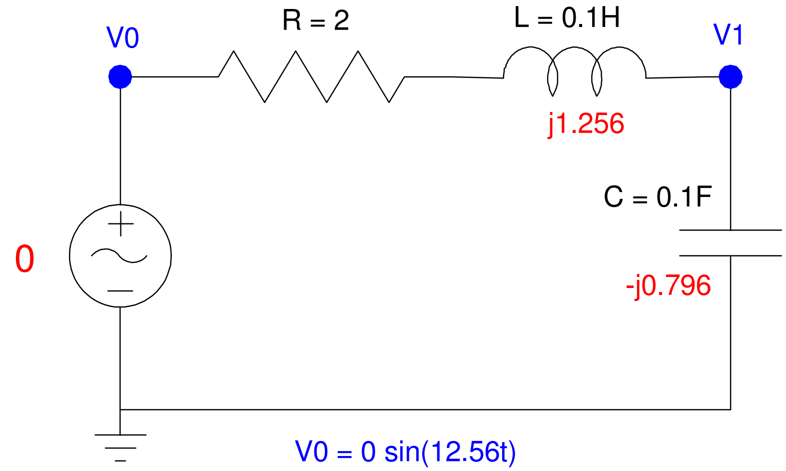


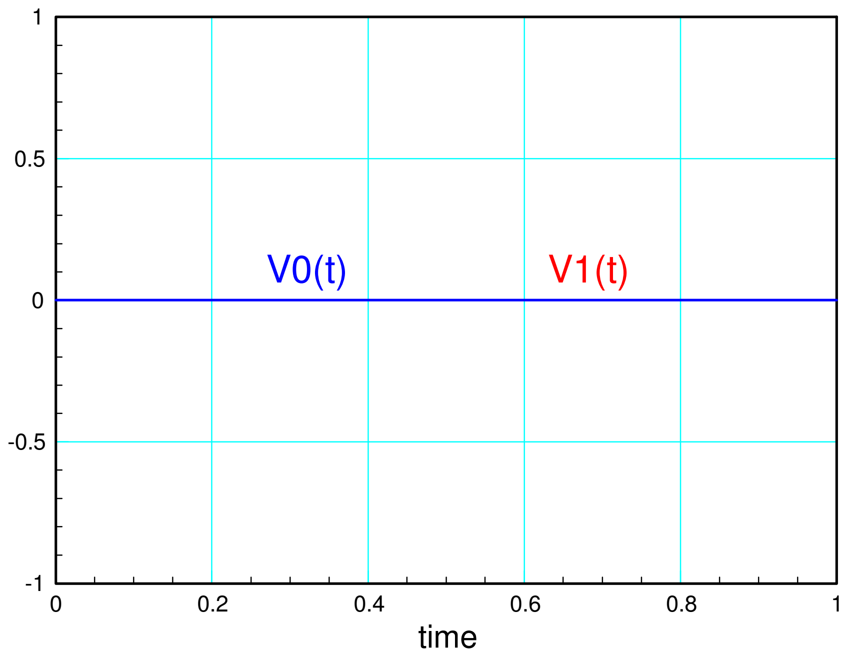
2Hz Analysis

$$v_0(t) = 0$$

The input is zero at 2Hz

- Output at 2Hz is zero





3Hz Analysis

$$v_0(t) = 0.2122\sin(6\pi t)$$

In phasor form

$$\omega = 6\pi$$

$$V_0 = 0 - j0.2122$$

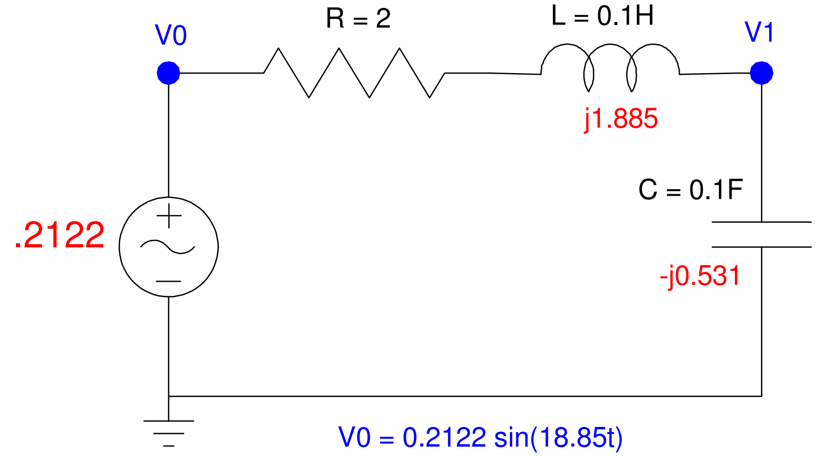
V1 is then

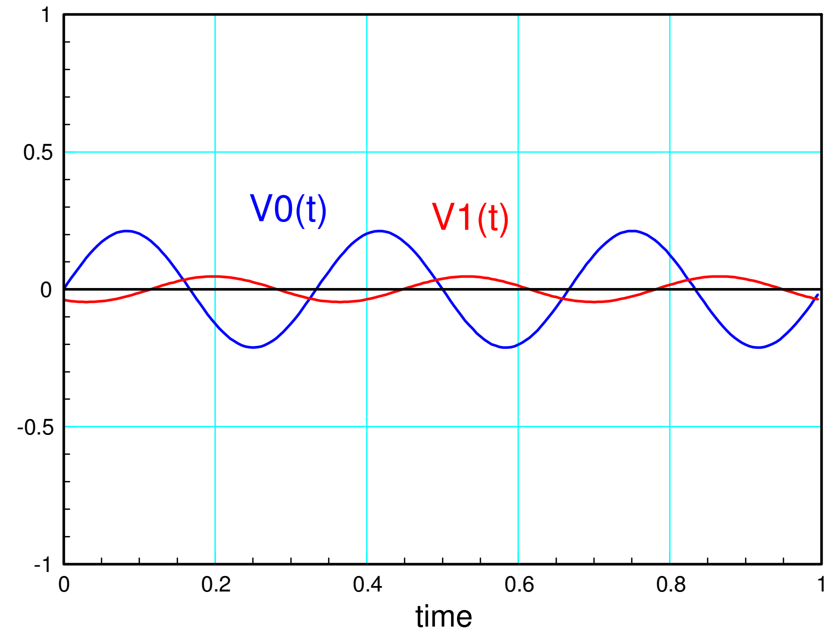
$$V_1 = \left(\frac{2}{(j0.04\omega)^2 + (j0.04\omega) + 1}\right)_{\omega=6\pi}(-j0.2122)$$

$$V_1 = -0.0386 + j0.0261$$

meaning

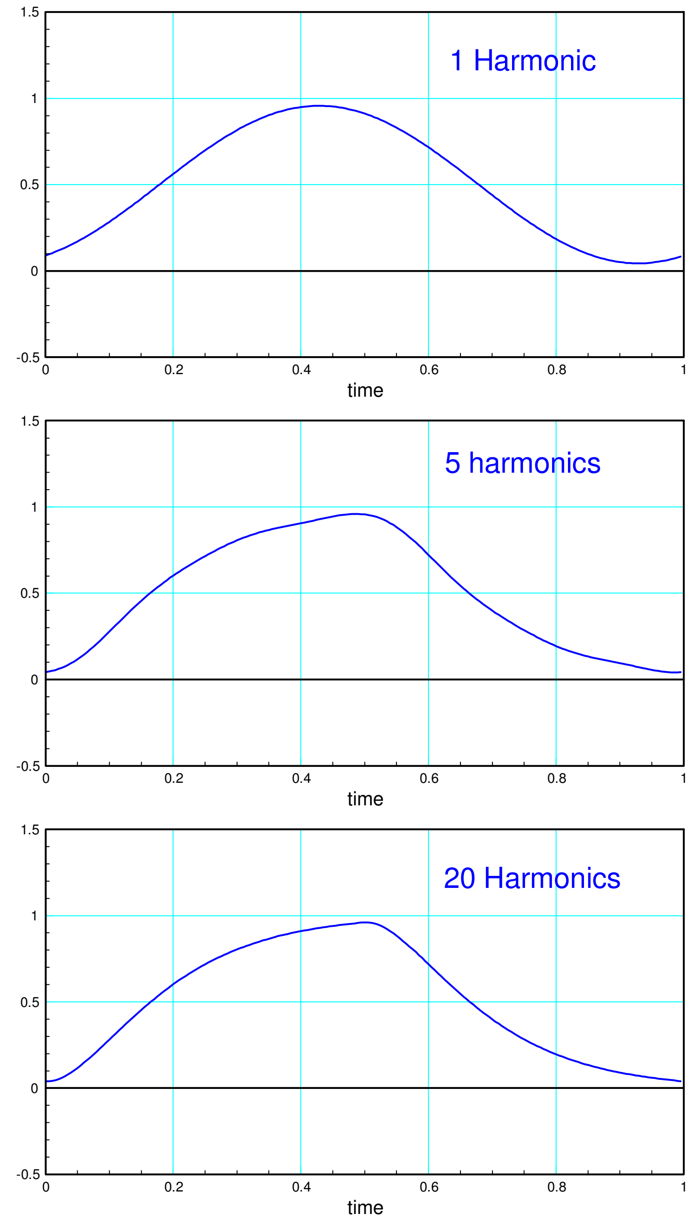
$$v_1(t) = -0.0386\cos(6\pi t) - 0.0261\sin(6\pi t)$$





Final Answer

$$v_1(t) = 0.5$$
$$-0.340\cos(2\pi t) + 1.268\sin(2\pi t)$$
$$-0.0386\cos(6\pi t) - 0.0261\sin(6\pi t)$$



Add them all together and you get V2(t)

- The more terms you add,
- The more accurate the response

Summary:

Once you have

- Phasors, and
- Superposition

you have Fourier Transforms.

Fourier Transforms allow you to solve differential equations with *any* periodic input

- Express x(t) as the sum of a DC term and harmonics
- Find y(t) at each frequency separately
- Add the results together to get the total y(t)

Matlab makes this process fairly easy

- A little tedious if you include lots of terms
- Typically, only a few terms are sufficient