Thevenin Equivalents & Max Power Transfer

ECE 211 Circuits I

Lecture #27

Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

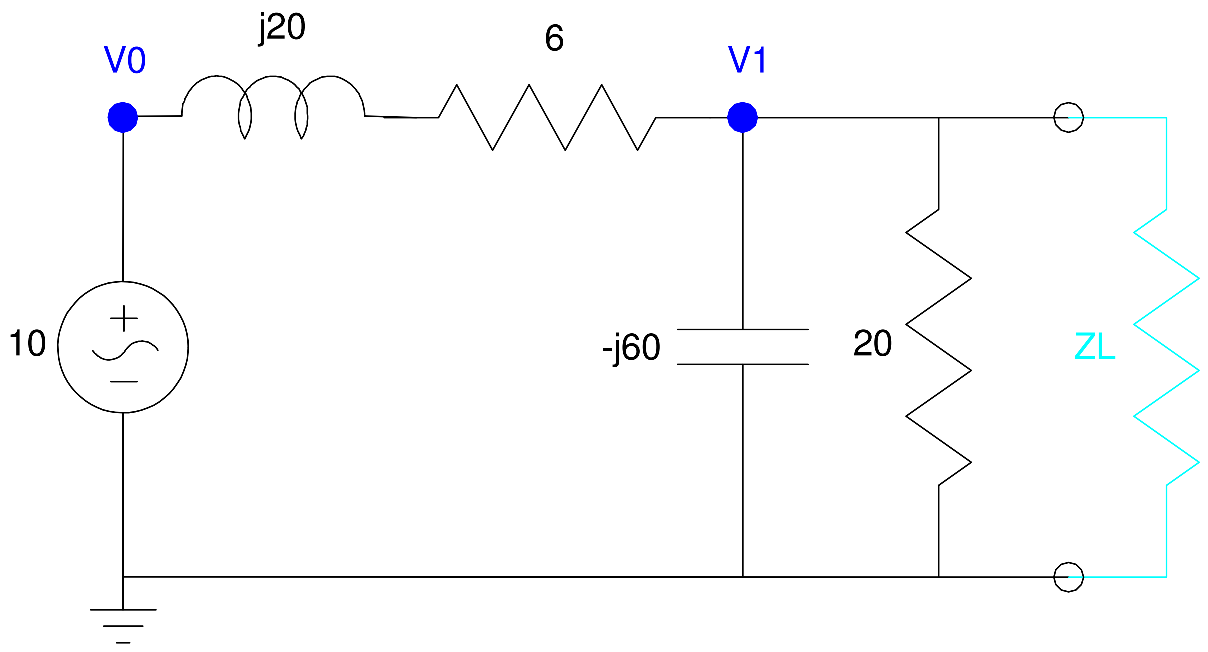
Thevenin Equivalents

Thevenin equivalents also work with phasors - only you get complex numbers for the Thevenin voltage and Thevenin resistance.

	VI relationship	Phasor Notation
Voltage	$v(t) = a\cos(\omega t) + b\sin(\omega t)$	$V = a - jb$
Resistor	$v = iR$	$Z_R = R$
Inductor	$v = L\frac{di}{dt}$	$Z_L = j\omega L$
Capacitor	$i = C\frac{dv}{dt}$	$Z_C = \frac{1}{j\omega C}$

Example 1: Determine

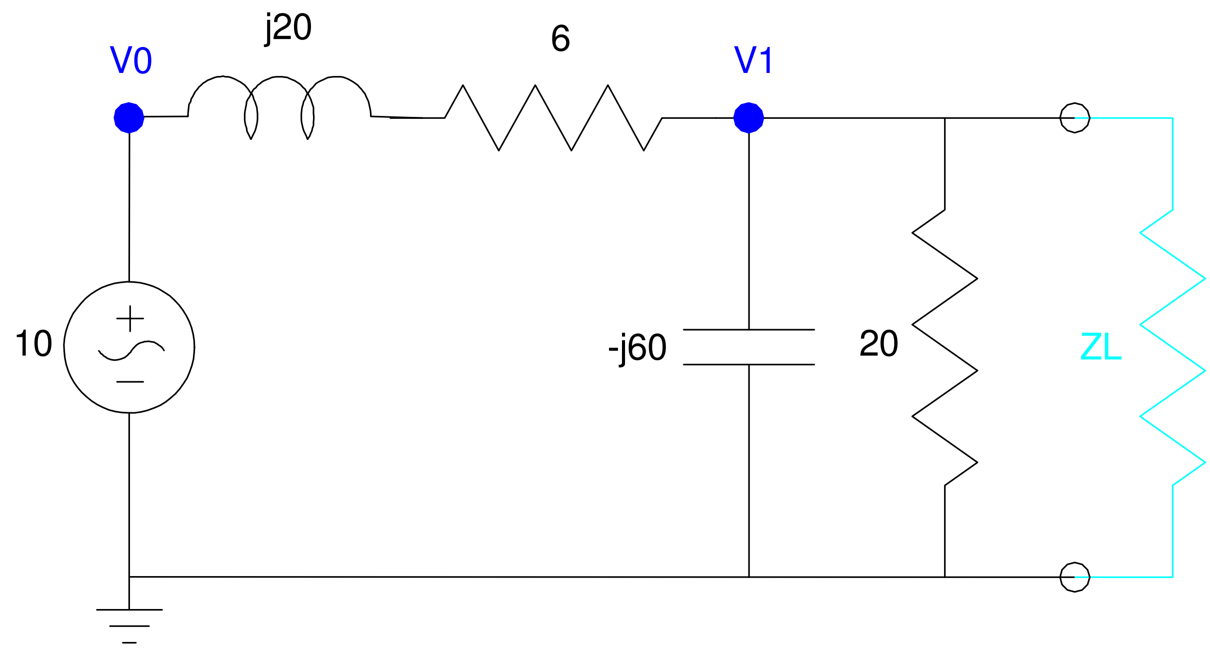
- The Thevenin equivalent for the following circuit,
- ZL for max power transfer, and
- The maximum power to a load



Thevenin Equivalent

Find two of the following:

- Open-circuit voltage
- Short-circuit current
- Equivalent resistance



Open-Circuit Voltage

Voltage Nodes

$$\left(\frac{V_1-V_0}{6+j20}\right)+\left(\frac{V_1}{-j60}\right)+\left(\frac{V_1}{20}\right)=0$$

Voltage division

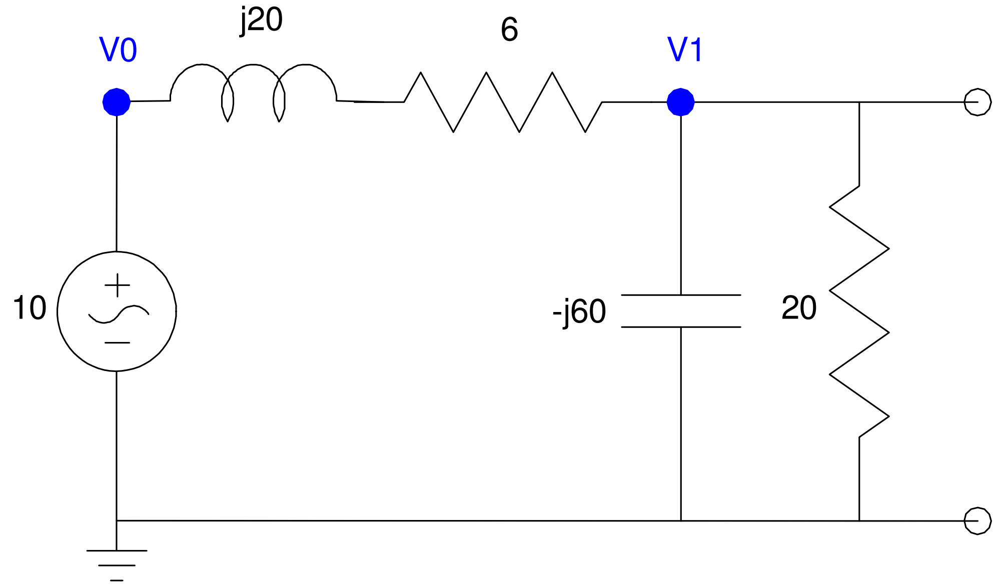
$$-j60||20=18-j6$$

$$V_1=\left(\frac{(18-j6)}{(18-j6)+(6+j20)}\right)10$$

Result

$$V_1=4.508-j5.129$$

This is Vth



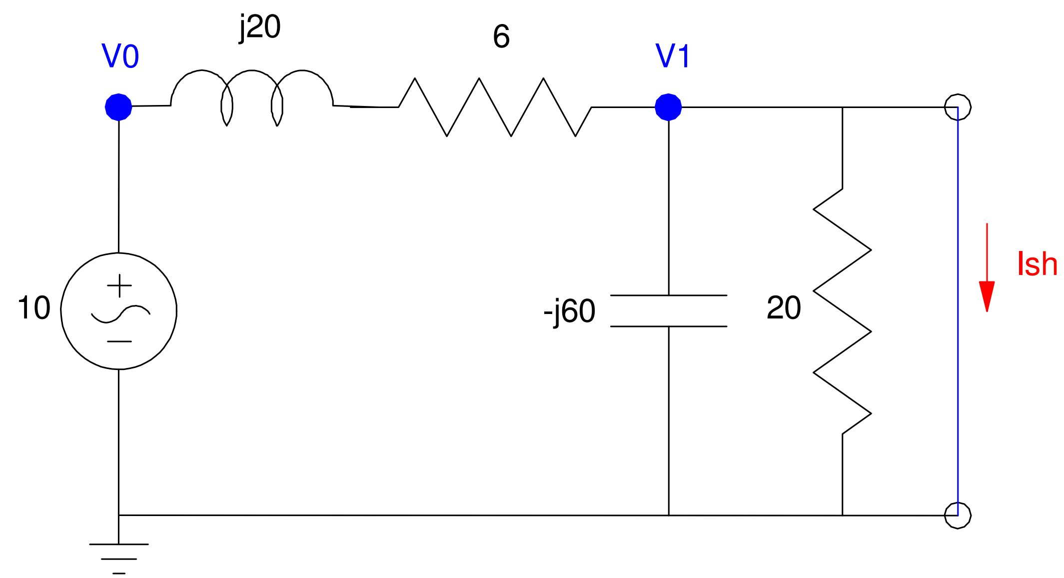
Short-Circuit Current

- Short the output
- Compute the current

$$I_{sh} = \left(\frac{10}{6+j20}\right)$$

$$I_{sh} = 0.138 - j0.459$$

This is I(Norton)



Thevenin Impedance

Option 1: Zth = V(open) / I(short)

$$Z_{th} = \left(\frac{4.5078 - j5.1295}{0.1376 - j0.4587}\right)$$

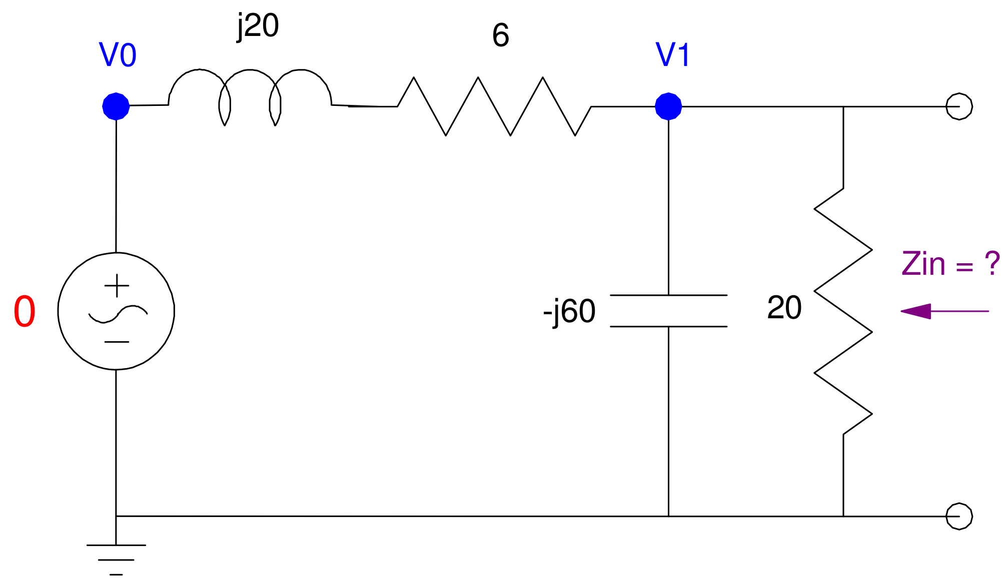
$$Z_{th} = 12.9637 + j5.9378$$

Option 2:

- Turn off the source
- Compute the impedance looking in

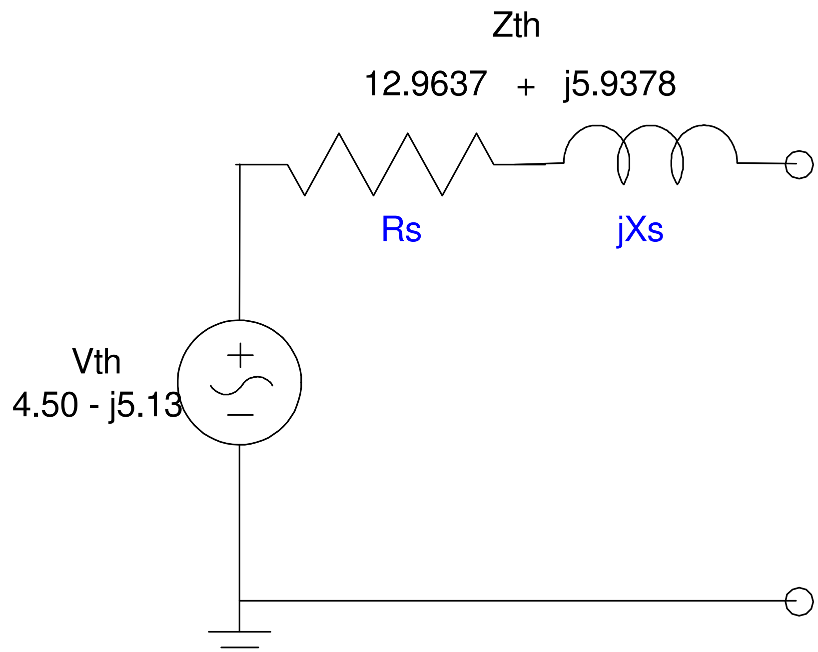
$$Z_{in} = (6 + j20) || (-j60) || (20)$$

$$Z_{in} = 12.9637 + j5.9378$$



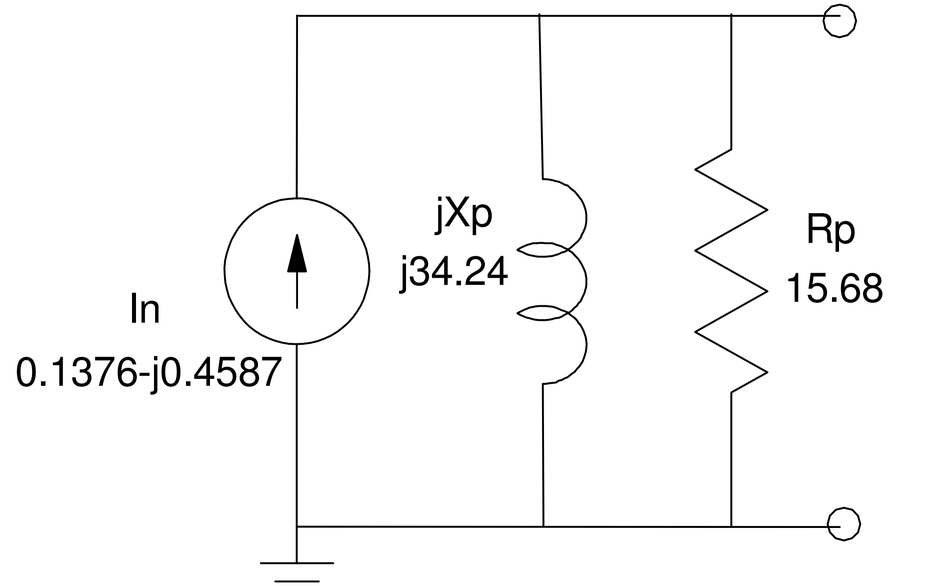
Thevenin Equivalent

- Vth is the open-loop voltage
- Zth is the Thevenin resistance
  - Series model
  - Rs + jXs



Norton Equivalent

- In is the short-circuit current
- Zn = Zth
  - Parallel model
  - Rp || jXp



Note

$$R_s + jX_s = R_p || jX_p$$

$$(R_s + jX_s) = \left(\frac{1}{R_p} + \frac{1}{jX_p}\right)^{-1}$$

$$\left(\frac{1}{R_s + jX_s}\right) = \frac{1}{R_p} + \frac{1}{jX_p}$$

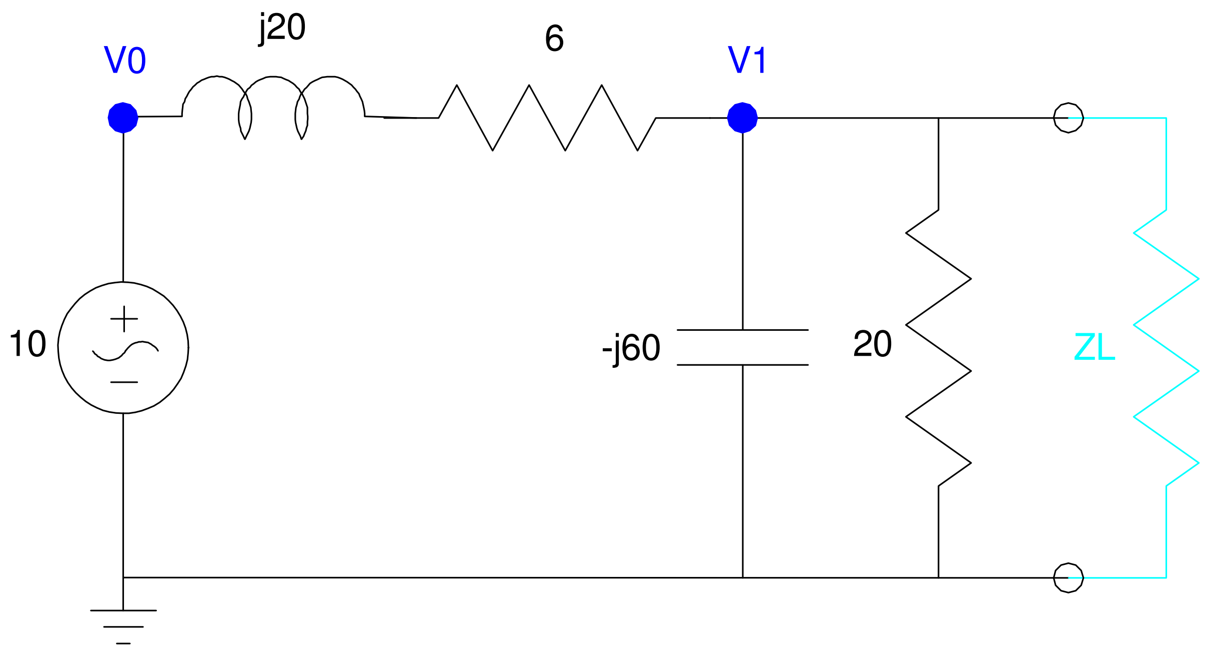
$$12.9637 + j5.9378 = (15.6835) || (j34.2408)$$

Max Power Transfer

- New Topic

What value of ZL maximizes the power to the load?

What is the maximum power you can transfer?



Power Calculations

Equations for power are slightly different DC vs. AC

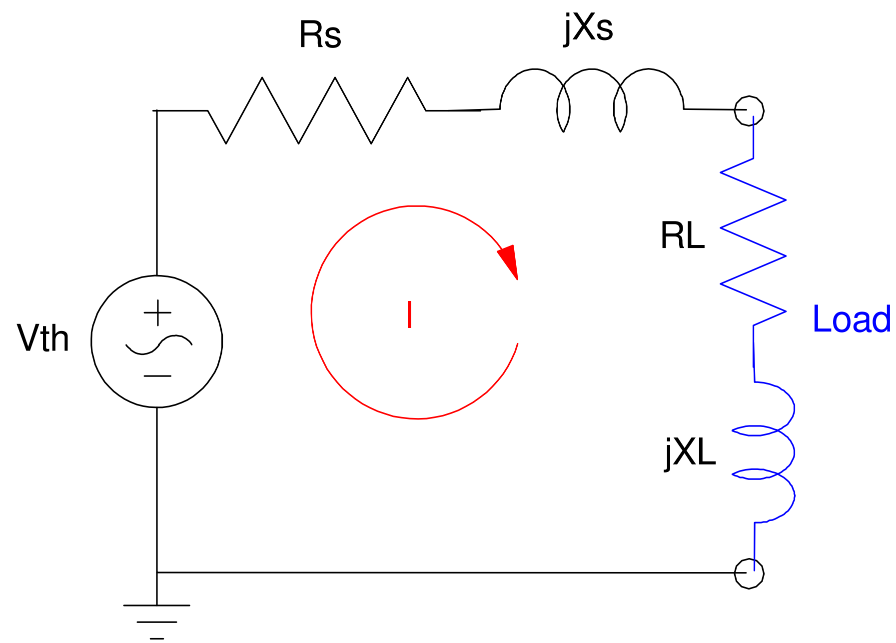
- The real part of P is the work done (or heat produced),
- The complex part of P is the energy that bounces back and forth.

DC	AC rms	AC peak
$P = V \cdot I$	$P = V_{rms} \cdot I_{rms}^*$	$P = \frac{1}{2} V_p \cdot I_p^*$
$P = I^2 R$	$P = \lvert I_{rms} \rvert^2 \cdot Z$	$P = \frac{1}{2} \cdot \lvert I_p \rvert Z$
$P = \frac{V^2}{R}$	$P = \frac{\lvert V_{rms} \rvert^2}{Z^*}$	$P = \frac{1}{2} \cdot \frac{\lvert\lvert V_{rms} \rvert\rvert^2}{Z^*}$

Maximum Power to the Load

Procedure

- Find the Thevenin equivalent
- Find the power to the load
- Maximize the real part



Calculations

Voltage (rms units)

$$V_L = \left(\frac{R_L + jX_L}{(R_L + jX_L) + (R_s + jX_s)}\right) V_{th}$$

Current (rms units)

$$I_L = \left(\frac{V_{th}}{(R_L + jX_L) + (R_s + jX_s)}\right)$$

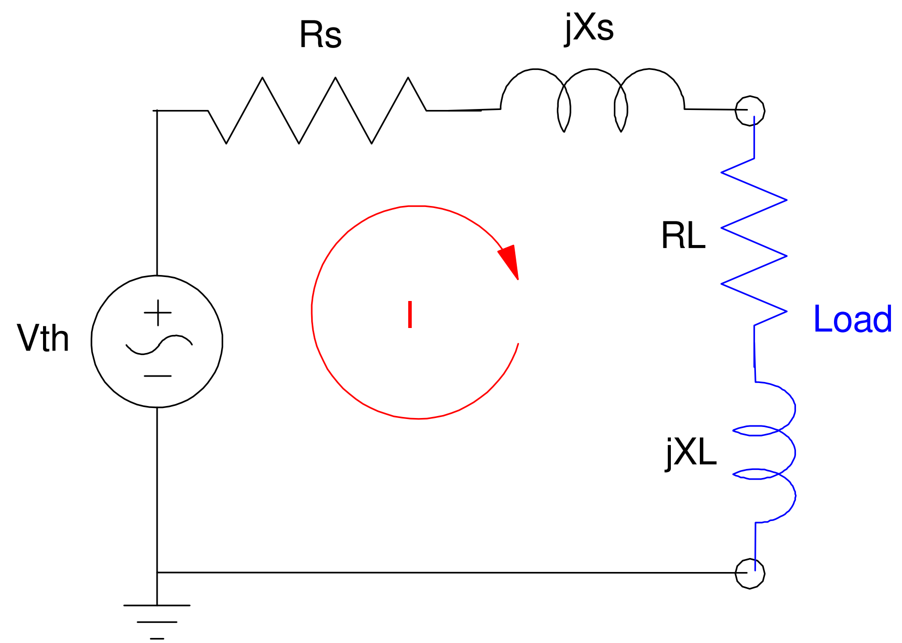
Power

$$P = V_L \cdot I_L^*$$

$$P = \left(\frac{R_L + jX_L}{(R_L + R_s)^2 + (X_L + X_s)^2}\right) |V_{th}|^2$$

Real part of power

$$P = \left(\frac{R_L}{(R_L + R_s)^2 + (X_L + X_s)^2}\right) |V_{th}|^2$$



Maximize power with respect to jXL

$$P = \left(\frac{R_L}{(R_L+R_s)^2+(X_L+X_s)^2}\right)|V_{th}|^2$$

$$\frac{dP}{dX_L} = 0 = \left(\frac{-R_L(2)(X_L+X_s)}{\left((R_L+R_s)^2+(X_L+X_s)^2\right)^2}\right)$$

$$X_L = -X_s$$

Make the complex impedance the negative of the complex part of the Thevenin series impedance

Maximize power with respect to RL

$$P = \left(\frac{R_L}{(R_L+R_s)^2+(X_L+X_s)^2}\right)|V_{th}|^2$$

$$P = \left(\frac{R_L}{(R_L+R_s)^2}\right)|V_{th}|^2$$

Maximize

$$\frac{dP}{dR_L} = 0 = \left(\frac{(R_L+R_s)^2-R_L(2)(R_L+R_s)}{(R_L+R_s)^2}\right)|V_{th}|^2$$

$$(R_L+R_s)^2 - R_L(2)(R_L+R_s) = 0$$

$$(R_L+R_s)(R_s-R_L) = 0$$

Solution:

$$R_L = R_s$$

Net Result

$$Z_L = Z_{th}^*$$

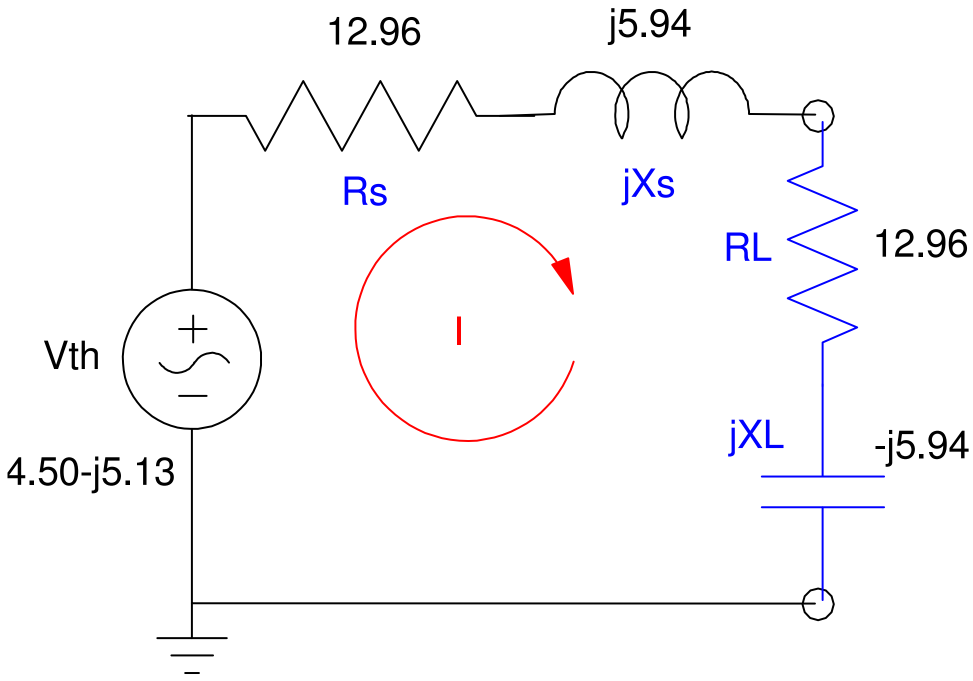
$$R_L + jX_L = R_s - jX_s$$

Max power is then

$$I = \frac{V_{th}}{R_s + R_L} = \frac{V_{th}}{2R_s}$$

$$P = V \cdot I^*$$

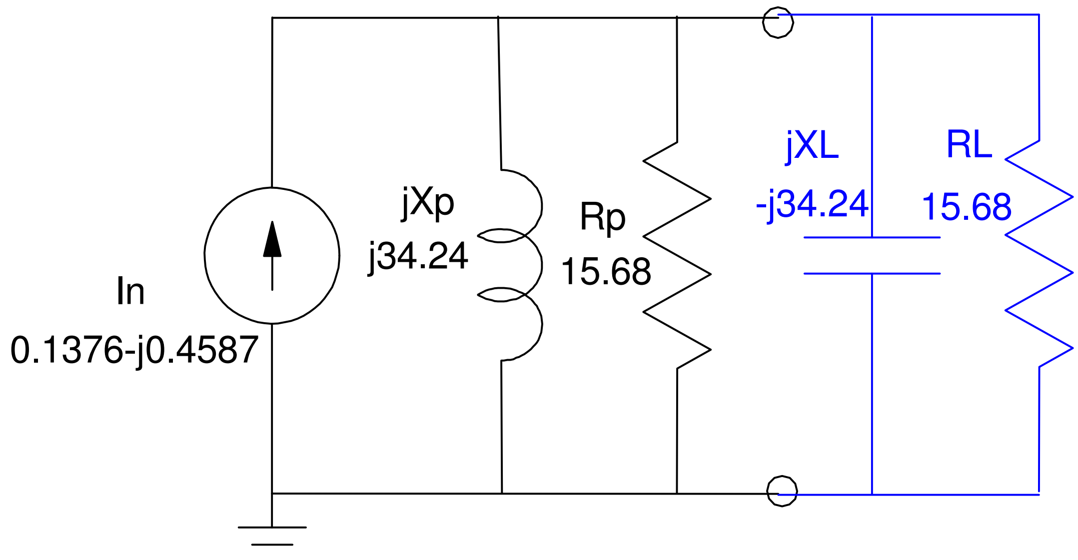
$$P = \frac{|V_{th}|^2}{2R_s}$$



Norton Model

You can also use the parallel model

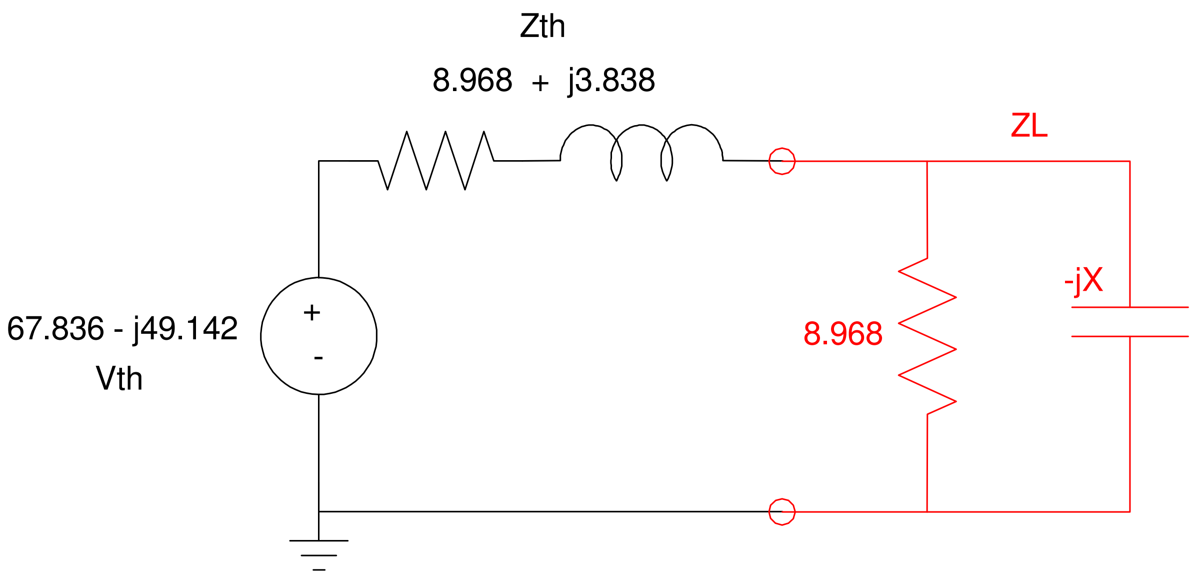
- jXL = -jXp
- RL = Rp



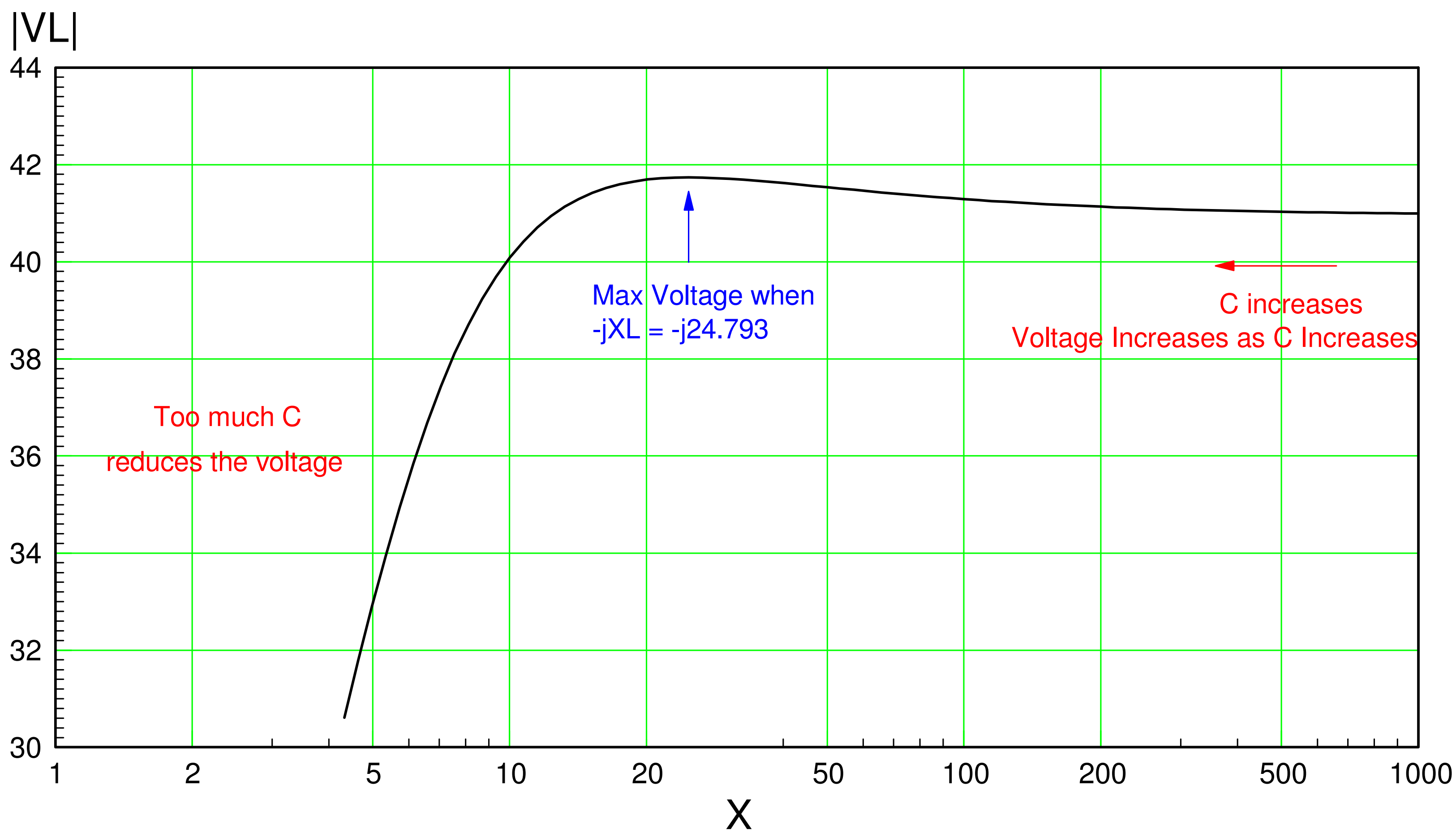
"Capacitors add Voltage"

If Zth is inductove, then adding capacitors to the load,

- Cancels the complex part of Zth, which
- Reduces the overall impedance, which
- Increases the current to the load, which
- Increases the voltage at the load.

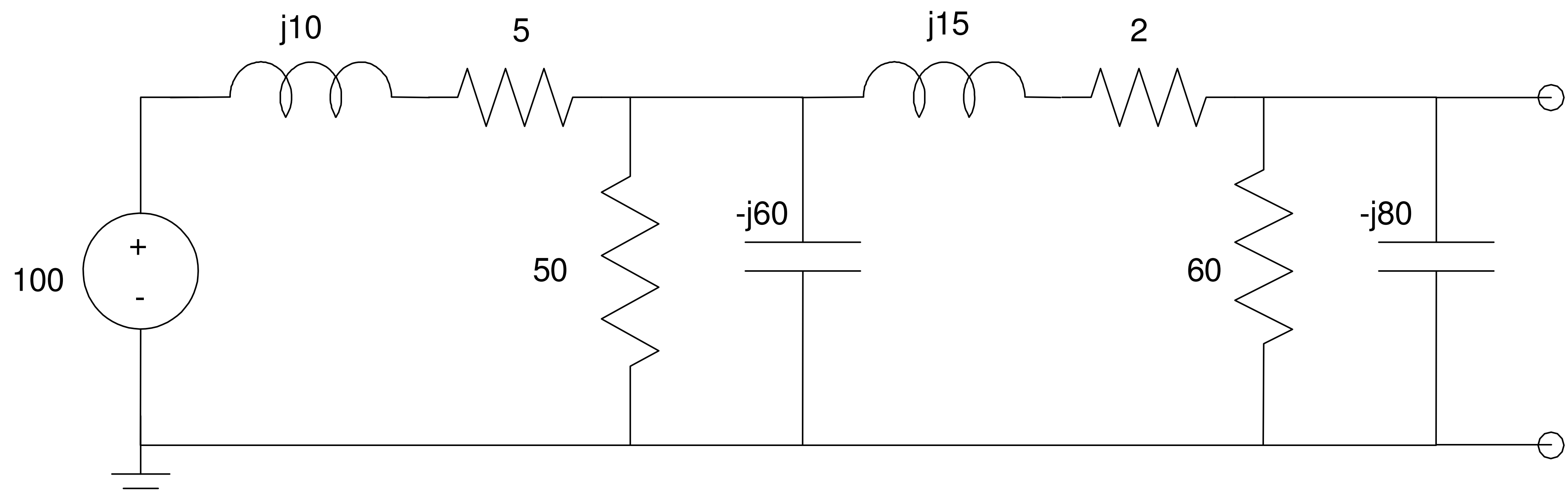


This only works up to a point: once you have cancelled all of the inductance (+jX), adding more capacitors will actually redice the voltage.



Example 3: Source Transformations

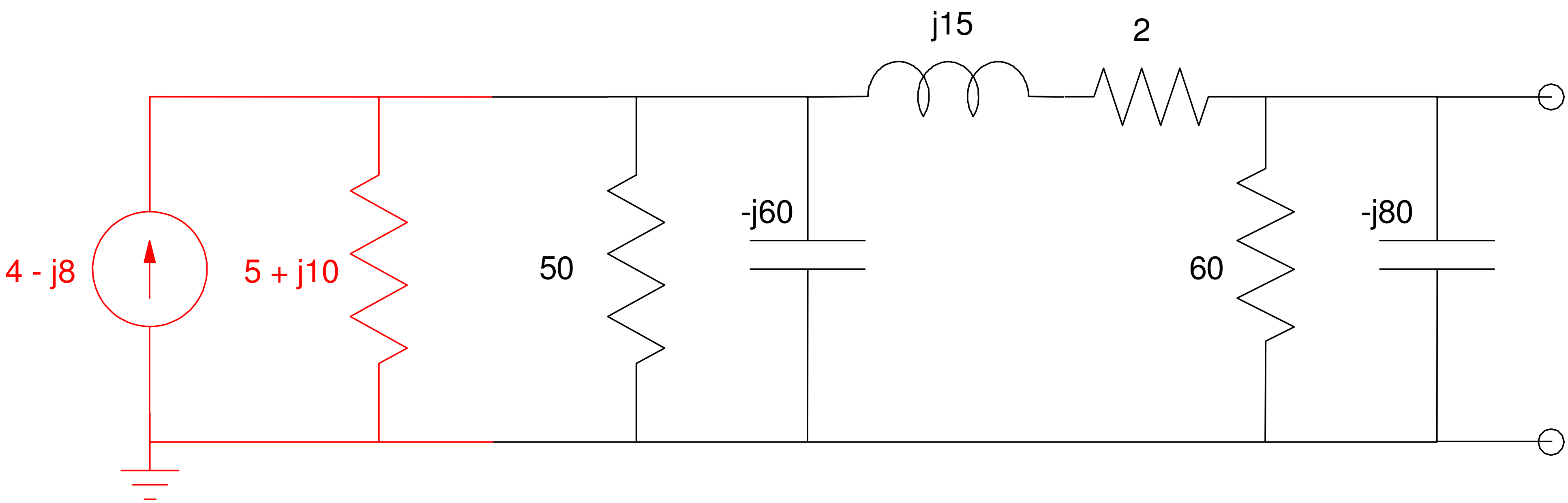
Source transformations also work with complex numbers. For example, determine the Thevenin equivalent for the following circuit:



Step 1: Convert to a Norton equivalent

$$Z_N = Z_{th} = 5 + j10$$

$$I_N = \frac{V_{th}}{Z_{th}} = 4 - j8$$



Combine impedances in parallel

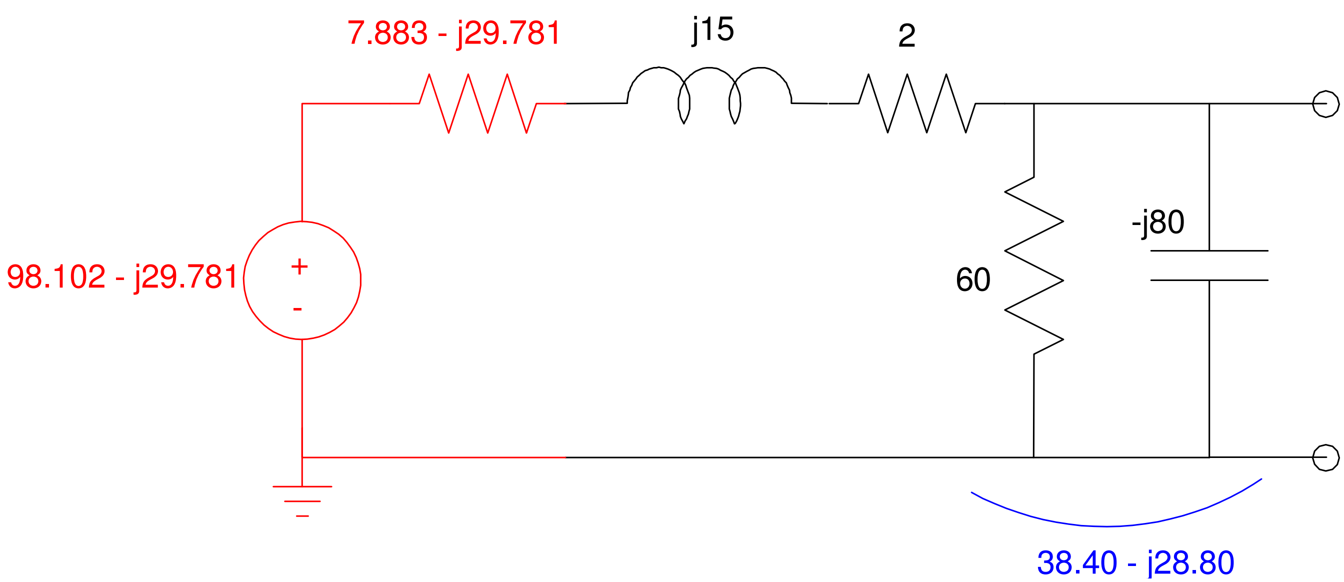
$$(5+j10)||(50)||(-j60) = 7.883 + j8.321$$

Convert to Thevenin

$$Z_{th} = Z_N = 7.883 + j8.321$$

$$V_{th} = I_N \cdot Z_N = (4-j8)\cdot(7.883+j8.321)$$

$$V_{th} = 98.102 - j29.781$$



Now find the Thevenin equivalent.

By voltage division:

$$V_{th} = \left(\frac{(38.40 - j28.80)}{(38.40 - j28.80) + (7.883 - j29.781) + (2 + j15)}\right)(98.102 - j29.781)$$

$$V_{th} = 68.701 - j74.405$$

Zth: Turn off the source and measure the impedance

$$Z_{th} = ((7.83 - j29.781) + (2 + j15))||(60)||(-j80)$$

$$Z_{th} = 20.077 + j14.931$$

Summary

RLC circuits can be reduced to their Thevenin and Norton equivalent

- Same procedure as DC
- Using phasors and complex numbers

Max power transfer is when

- Series Connection
  - $R_L = Rs$
  - $jX_L = -jXs$
- Parallel Connection
  - $R_L = Rp$
  - $jX_L = -jXp$