The p-Operator

Background

Differential equations describe circuits which contain inductors and/or capacitors. The basic VI relationship for these two are

$$V = L\frac{dI}{dt}$$

$$I = C\frac{dV}{dt}$$

Likewise, being able to solve differential equations is fundamental to circuit analysis when inductors and capacitors are involved.

If you have a differential equation with a forcing function that turns on and off, the output typically contains two parts:

- The transient response, and
- The steady-state response

For example, if you apply a 0V or 5V step input to a differential equation

$$y' + 10y = 7x$$

the steady-state solution is either 0V or 3.5V (steady-state response). The path to get to steady-state is the transient response as shown in Figure 1.

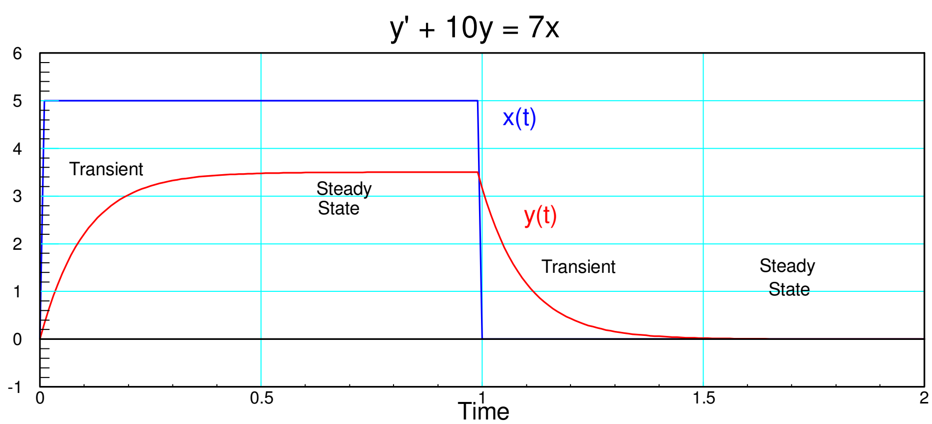

Figure 1: Solutions to differential equations typically have a transient response then a steady-state response

So far, we've only looked at finding the steady-state solution. This is what phasors do. In the next few lectures, we'll look at how to find the transient response.

Methods for Finding the Transient Response

There are several methods for finding the transient response.

- In Calculus I - III, you typically guess the form of the answer. You then take derivatives and try to make the equations balance.

- In Circuits I and II, CircuitLab is one of the methods used. With CircuitLab, you input the circuit and run a time-domain simulation.
- The p-operator is another method - which is introduced in Circuits I.
- LaPlace transforms are an extension of the p-operator and are covered in Circuits II and Signals and Systems.

Calculus I-III Method

Assume you have a differential equation along with a forcing function (x) and an initial condition. For example, assume

$$\frac{dy}{dt} + 3y = 4x$$

$$x(t) = 5e^{-2t}$$

$$y(0) = 3$$

Find y(t) for t>0.

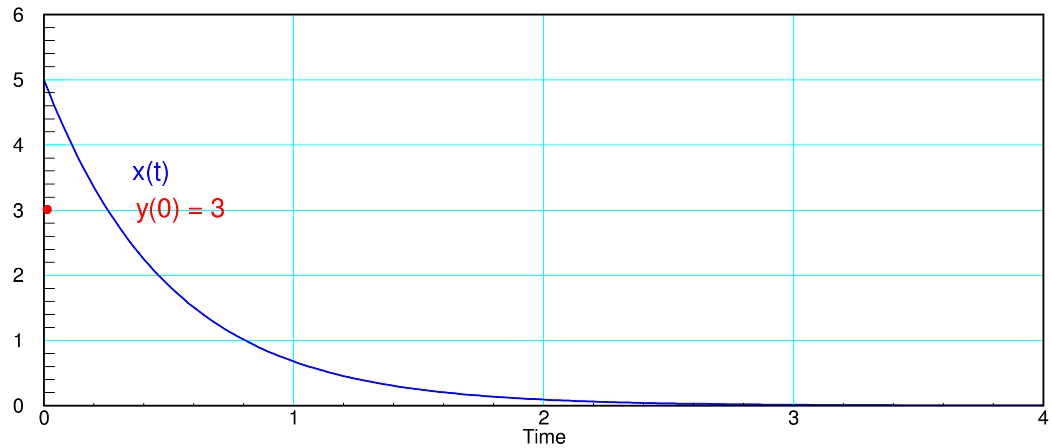

Figure 2:

Using superposition, y(t) will contain two terms:

- y(t) assuming that x(t) = 0 (the natural response), and
- y(t) assuming that x(t) was on for all time (the forced response).

Start with the forced response. Plugging in x(t) into the differential equation gives

$$\frac{dy}{dt} + 3y = 4(5e^{-2t})$$

In calculus, you guess the form of the solution. Guessing that y(t) is of the form

$$y(t) = ae^{-2t}$$

Take the derivative and substitute for y(t) and dy/dt

$$(-2ae^{-2t}) + 3(ae^{-2t}) = 20e^{-2t}$$

Solve for 'a'

$$ae^{-2t} = 20e^{-2t}$$

$$a = 20$$

Now solve for the natural response. Assuming x(t) = 0 gives the differential equation

$$\frac{dy}{dt} + 3y = 0$$

From experience, y(t) must be of the form

$$y(t) = be^{-3t}$$

Some lucky guessing or lots of experience is needed to guess the right form of y(t).

The total response is then the forced response plus the natural response

$$y(t) = 20e^{-2t} + be^{-3t}$$

Plug in the initial conditions

$$y(0) = 3 = 20 + b$$

$$b = -17$$

and you have the total response

$$y(t) = 20e^{-2t} - 17e^{-3t}$$

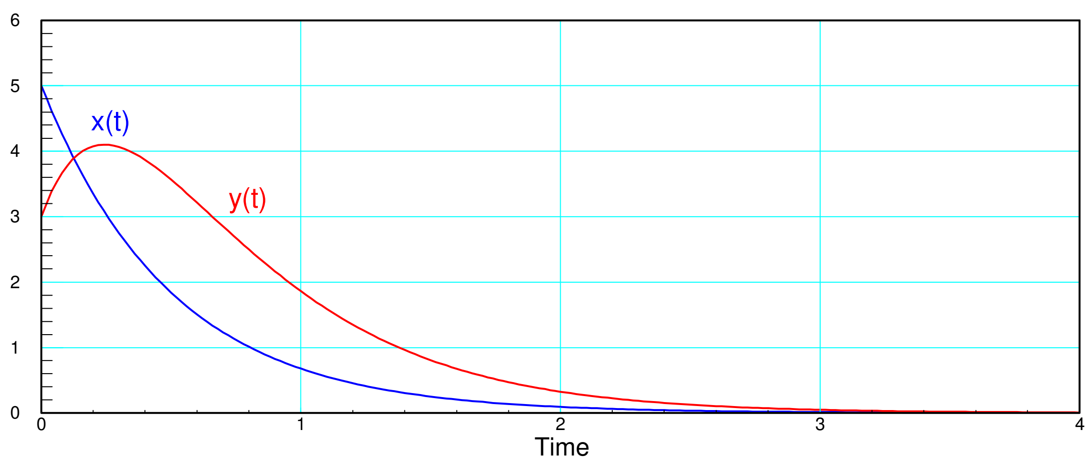

Figure 3: Total response y(t)

Note that this method works, but it involves a lot of guesswork and is rather tedious. Even for a simple first-order differential equation. We need a better tool. That's the p-operator.

The p-Operator

The p-perator is a trick that turns differential equations into algebraic equations in p. The assumption is that algebra is easier than calculus. The basic assumption behind the p-operator is that all functions are of the form

$$x(t) = e^{pt}$$

With this assumption, differentiation becomes multiplication by 'p'

$$\frac{dx}{dt} = pe^{pt} = px$$

This in turn lets express differential equations in terms of transfer functions.

For example, assume x and y are related by the following differential equation

$$\frac{dy}{dt} + 3y = 4x$$

Using the p-operator, this becomes

$$pY + 3Y = 4X$$

or

$$Y = \left(\frac{4}{p+3}\right)X = G(p) \cdot X$$

The term G(p) acts like a gain from X to Y. This is called *the transfer function*.

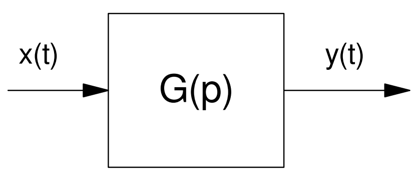

Figure 4: The transfer acts like a gain from X to Y

With the transfer function, the forced response for different inputs can be found fairly easily.

Case 1: x(t) = 5

This is really a phasor problem. X and Y are related by the transfer function

$$Y = \left(\frac{4}{p+3}\right)X$$

Along with the forcing function and the initial condition:

$$x(t) = 5$$

$$y(0) = 3$$

The assumption behind the p-operator is that x(t) is of the form

$$x(t) = ae^{pt}$$

Matching terms results in a=5 and p=0

$$5 = a \cdot e^{pt}$$

$$5 = 5 \cdot e^{0t}$$

The forced response is simply gain times input

$$Y = \left(\frac{4}{p+3}\right)X$$

$$Y = \left(\frac{4}{p+3}\right)_{p=0} \cdot (5)$$

$$Y = \frac{20}{3}$$

The natural response comes from assuming X=0

$$Y = \left(\frac{4}{p+3}\right) \cdot 0$$

$$(p+3)Y = 0$$

This means that either

Y = 0

(the trivial solution), or

p = -3

The latter implies that y(t) is of the form

$$y(t) = be^{-3t}$$

Putting the two together gives the total response

$$y(t) = \frac{20}{3} + be^{-3t}$$

Plug in the initial condition

$$y(0) = 3 = \frac{20}{3} + b$$

$$b = -\frac{11}{3}$$

and you get y(t)

$$y(t) = \frac{20}{3} - \frac{11}{3}e^{-3t}$$

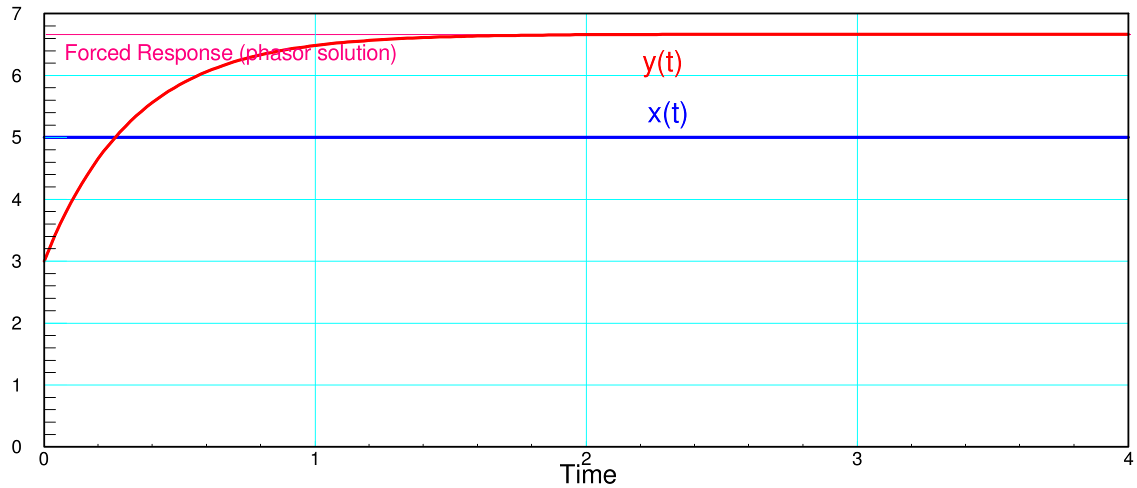

Figure 5: Total response (forced plus natural) for a constant input

Case 2: Exponential Input

Next, consider the case where x(t) is a decaying exponential

$$y' + 3y = 4x$$

$$x(t) = 5e^{-2t}$$

$$y(0) = 3$$

The forced response is really a phasor problem. Again, the basic assumption behind the p-operator is that all functions are in the form of

$$x(t) = a \cdot e^{pt}$$

Meaning

- a = 5
- p = -2

Going back to the transfer function

$$Y = \left(\frac{4}{p+3}\right)X$$

The forced response is

$$Y = \left(\frac{4}{p+3}\right)_{p=-2} \cdot (5)$$

$$Y = 20$$

meaning that the forced response is

$$y(t) = 20e^{-2t}$$

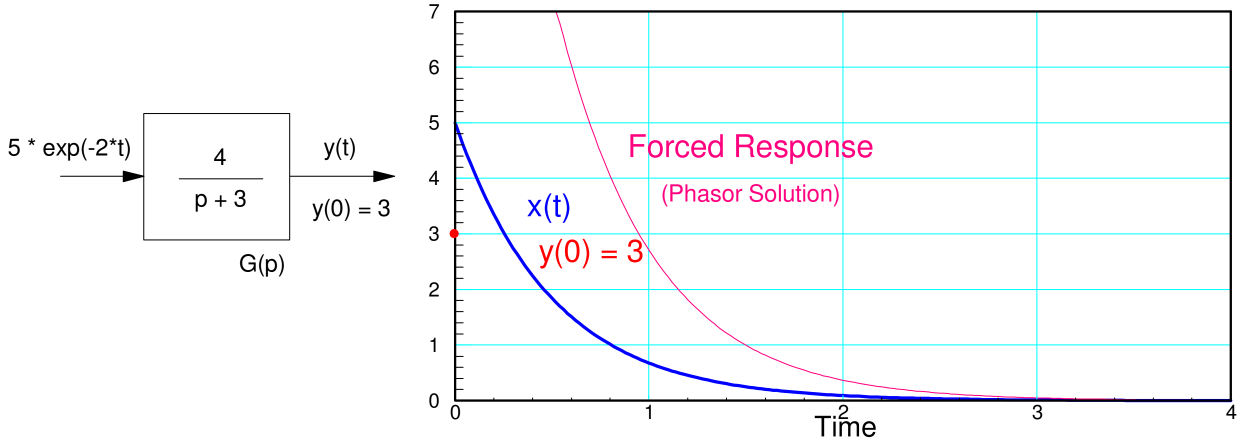

Figure 6: Forced response to an exponential input

The total response includes both the forced response and the natural response.

$$y(t) = 20e^{-2t} + be^{-3t}$$

Plug in the initial condition

$$y(0) = 3 = 20 + b$$

$$b = -17$$

giving

$$y(t) = 20e^{-2t} - 17e^{-3t} \qquad t > 0$$

Note that this is the same result we got using calculus - only a *lot* easier to compute.

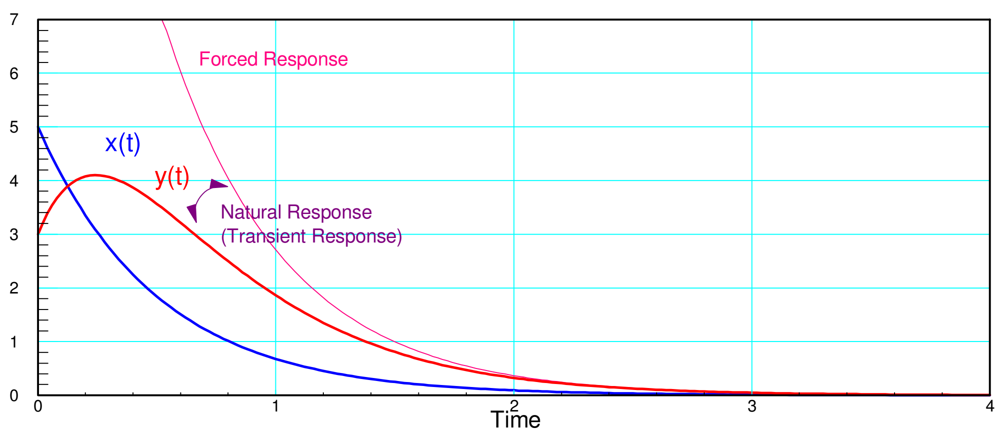

Figure 7: Total response (forced plus natural) for an exponential input

Case 3: Sinusoidal Input

Finally, consider the case where x(t) is a sinusoid

$$x(t) = 5\sin(2t)$$

$$y(0) = 3$$

From phasors, the steady-state response comes from

$$p = j2$$

$$X = 0 - j5$$

Y is then

$$Y = \left(\frac{4}{p+3}\right)X$$

$$Y = \left(\frac{4}{p+3}\right)_{p=j2} \cdot (0 - j5)$$

$$Y = 4.1654 - j3.0769$$

This is the phasor representation of

$$y(t) = 4.1654\cos(2t) + 3.0769\sin(2t)$$

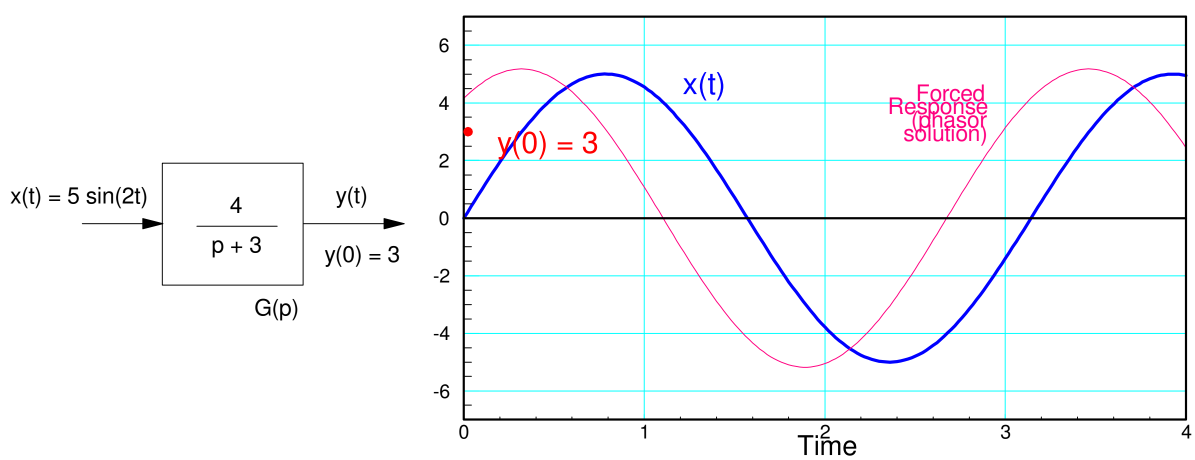

Figure 8: Stead-state (a.k.a. forced or phasor solution) for a sinusoidal input

The total response includes the natural response

$$y(t) = 4.1654\cos(2t) + 3.0769\sin(2t) + be^{-3t}$$

Plugging in the initial condition

$$y(0) = 3 = 4.1654 + b$$

$$b = -1.1654$$

y(t) is then

$$y(t) = 4.1654\cos(2t) + 3.0769\sin(2t) - 1.1654e^{-3t} \qquad t>0$$

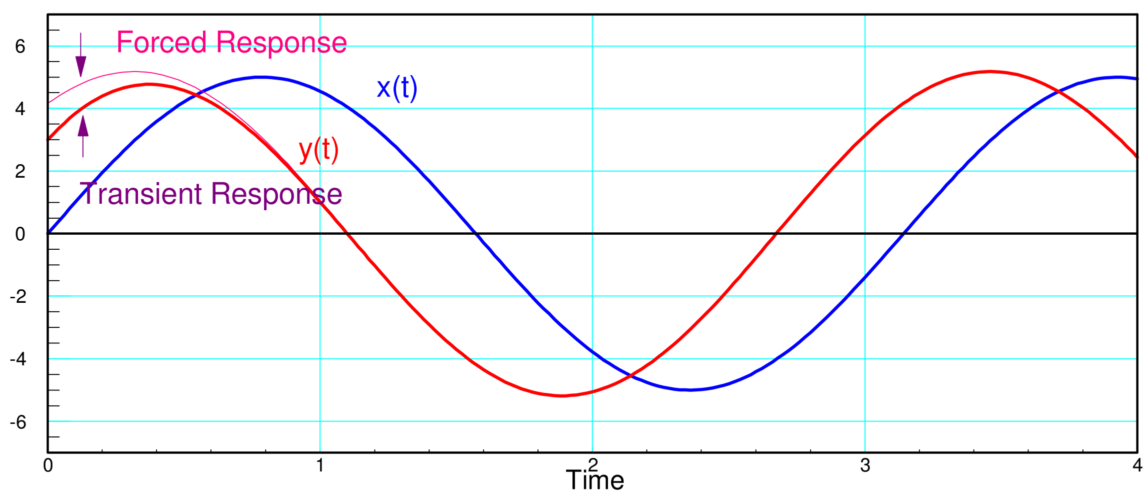

Figure 10: Total response including the forced response and the transient (natural) response

2nd-Order Differential Equations

The p-operator also works with second-order differential equations. In this case, however, two initial conditions are needed, such as y(0) and y'(0). For example, assume x and y are related by the following 2nd-order differential equation:

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 5y = 10x$$

Find y(t) assuming

$$x(t) = 3\cos(2t) + 4\sin(2t)$$

$$y(0) = y'(0) = 0$$

To do this, use the p-operator to convert this differential equation into a transfer function in p. Replacing derivatives with 'p' gives

$$p^2Y + 6pY + 5Y = 10X$$

or

$$Y = \left(\frac{10}{p^2+6p+5}\right) X$$

If x(t) is a 2 rad/sec sine wave

$$x(t) = 3\cos(2t) + 4\sin(2t)$$

then using phasors,

$$p = j2$$

$$X = 3 - j4$$

The forced response (a.k.a. steady-state response or phasor solution) is

$$Y = \left(\frac{10}{p^2+6p+5}\right)_{p=j2} \cdot (3 - j4)$$

$$Y = -3.1034 - j2.7586$$

meaning

$$y(t) = -3.1034\cos(2t) + 2.7586\sin(2t)$$

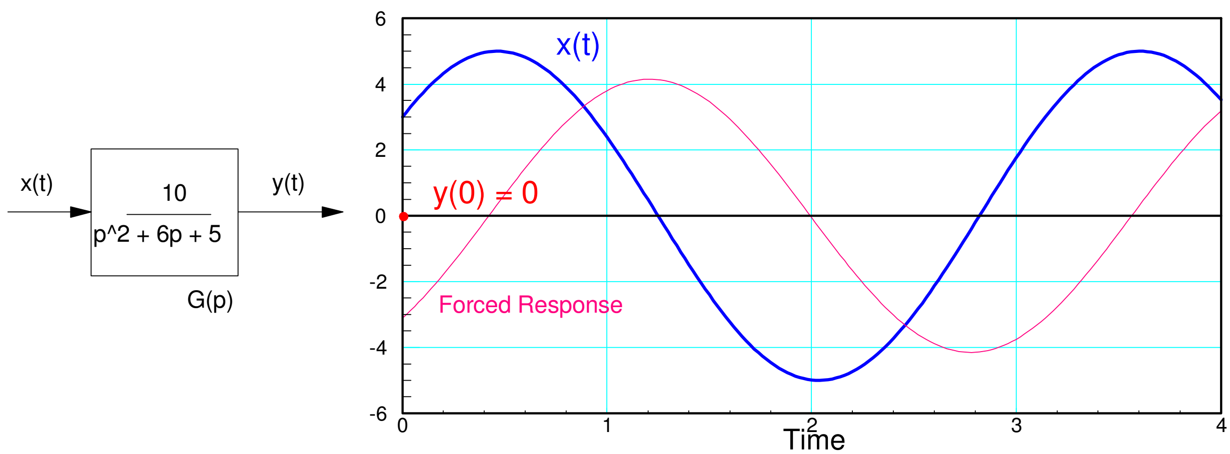

Figure 11: Forced response for a sinusoidal input (phasor solution)

The natural response is the response when x(t) = 0

$$Y = \left(\frac{10}{p^2+6p+5}\right) \cdot 0$$

$$(p^2 + 6p + 5)Y = 0$$

$$(p+1)(p+5)Y = 0$$

This has two solutions: the trivial solution

$$y(t) = 0$$

and

$$p = \{-1, -5\}$$

The latter implies the natural response is of the form

$$y(t) = ae^{-t} + be^{-5t}$$

Adding together the forced response to the natural response gives

$$y(t) = (-3.1034\cos(2t) + 2.7586\sin(2t)) + (ae^{-t} + be^{-5t})$$

Now, plug in the initial conditions

$$y(0) = 0 = -3.1034 + a + b$$

$$y'(0) = 0 = 2 \cdot 2.7586 - a - 5b$$

Solving (takes some algebra) gives

$$a = 2.5000$$

$$b = 0.6034$$

meaning the total response is

$$y(t) = (-3.1034\cos(2t) + 2.7586\sin(2t)) + (2.5e^{-t} + 0.6034e^{-5t})$$

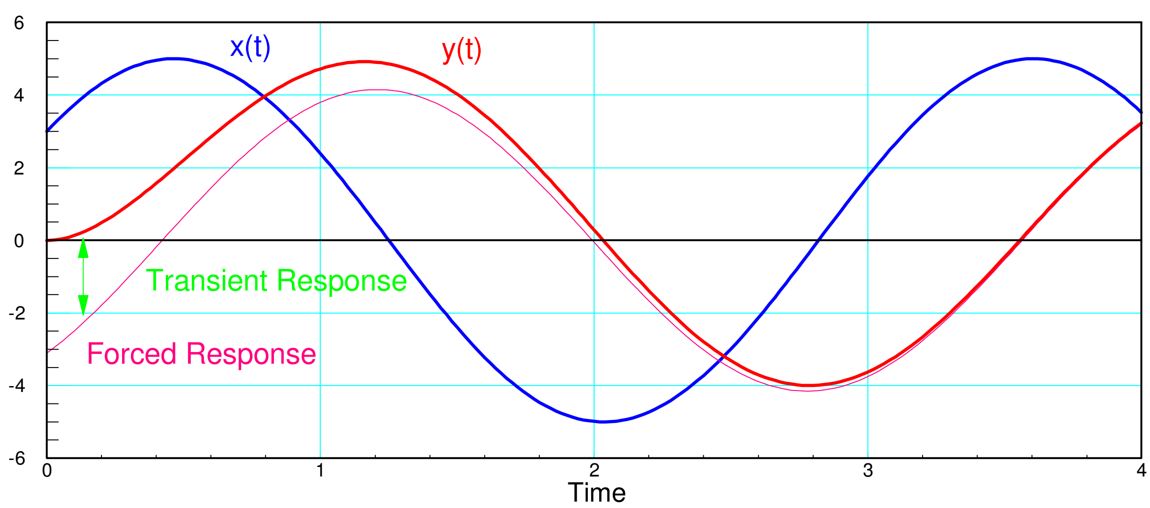

Figure 12: Total response including the forced response and the transient response

Summary

The p-operator assumes all functions are of the form

$$y(t) = e^{pt}$$

With this assumption, the p-opeator turns differential equations into algebraic equations in p. This lets you solve for the output in terms of a gain times the input. That gain is called *the transfer function.*

$$Y = G(p) \cdot X$$

In general, the solution to a differential equation includes both a forced response and a natural response. The forced response (also known as the steady-state solution) is found using phasors. The natural response is determined by the initial conditions as well as the roots of the denominator of G(p).