Superposition with Phasors

Introduction

Up to now, we've been covering how to find voltages and currents for RLC circuits

- With DC inputs, and
- With AC inputs

For both of these, the main trick is set up N equations to solve for N unknowns. With DC inputs, real numbers are used. With AC inputs, complex numbers and phasors are used.

In this lecture, we'll look at solving for voltages and currents in a circuit with multiple inputs such as shown in Figure 1. This includes

- A DC and an AC input, and
- With two AC inputs at different frequencies

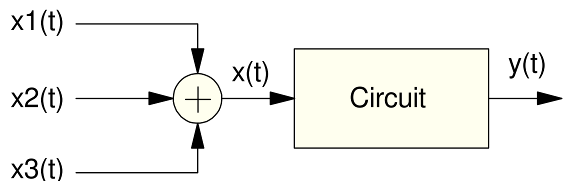

Figure 1: Find the output when there are multiple inputs at different frequencies

Linear Systems

One way to solve this sort of problem is to use the property of linearity. If a system is linear, then

$$f(x_1 + x_2 + x_3) = f(x_1) + f(x_2) + f(x_3)$$

What this means is that

- If a system has an input which is the sum of N different terms,
- You can treat this as N separate problems.

For each problem, find the output assuming that only that term is applied at the input. The total output will then be the sum of each individual output as shown in Figure 2. This is called *superposition*.

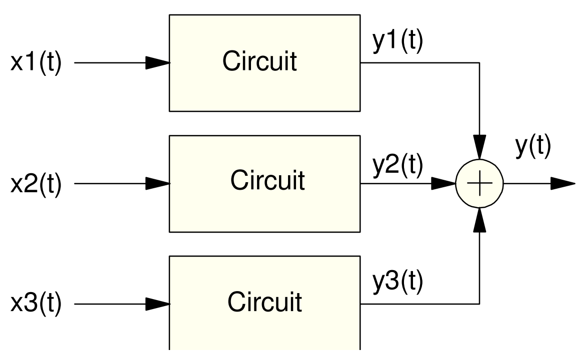

Figure 2: Treat this problem as the sum of three separate problems (superposition)

Case 1: DC + AC

Let's start with the circuit presented in Figure 3 where the input contains two terms: a DC term and an AC term. This is a common type of problem you'll encounter in electronics. One of the circuits you'll cover in electronics is AC to DC converter.

- The DC portion of V1 is the what you want: a DC signal that can drive a load.
- The AC portion of V1 represents the ripple: how much V1 varies about its constant value.

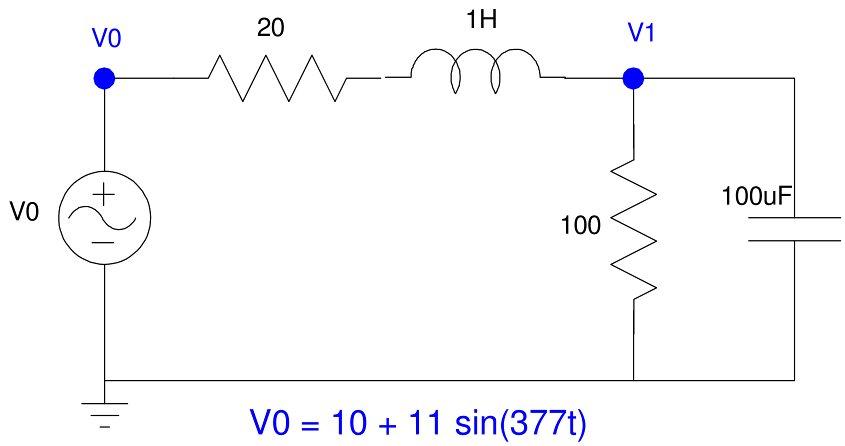

Figure 3: Find V1(t) when V0 contains a DC term and an AC term

CircuitLab: One way to find V1 is to use CircuitLab. When inputting V0, an AC source can be used. If you double-click on that source, a menu appears as shown in Figure 4. With this menu, you can

- Set the offset to 10V (the DC term for V0)
- Set the amplitude to 11V (the AC term for V0), and
- Set the frequency to 60Hz (377 rad/sec)

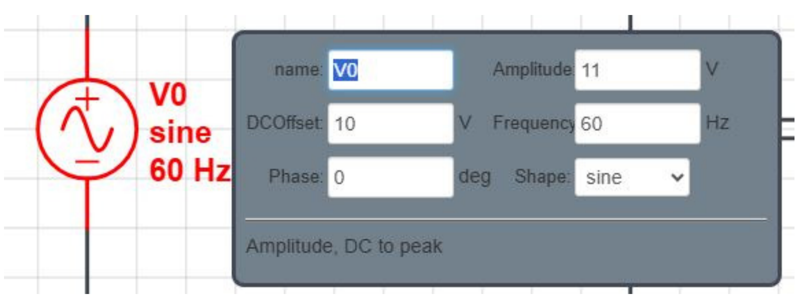

Figure 4: In CircuitLab, add a DC and AC term for V0

You can then run a *Time-Domain Simulation* to find V1(t).

- Start at 100ms to allow the transients to die out (phasor analysis only gives the steady-state solution),
- Run the simulation for 40ms (about 3 cycles at 60Hz), and
- Set the step size to 40us (gives 1000 points on the graph)

This will produce the signal V1 as shown in Figure 5.

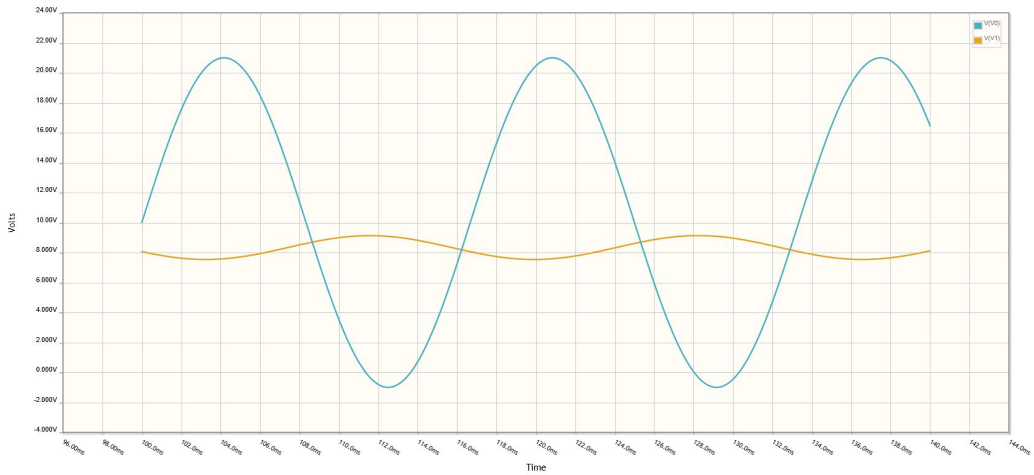

Figure 5: V0(t) (blue) and V1(t) (orange)

From this graph, you can see that V1 has a DC and an AC component

- The DC term is 8.332V (the average of Vmax and Vmin)
- The AC term is 1.591Vpp (the difference: Vmax - Vmin)

Note that for AC terms, units matter:

- V1 = 1.591Vpp
- V1 = 795.5mVp
- V1 = 562.5Vrms

The answer depends upon whether you are using peak-to-peak voltage, peak-voltage, or rms voltage. Peak-to-peak voltage is defined as:

$$V_{pp} = V_{\max} - V_{\min}$$

Peak voltage (or ground-to-peak) is

$$V_p = \frac{1}{2} V_{pp}$$

rms voltage (root-mean-squared) is

$$V_{rms} = \frac{1}{\sqrt{2}} V_p = \frac{1}{2\sqrt{2}} V_{pp}$$

The three are different. For AC signals, you need to label your answer (specify what kind of voltage you're talking about).

To analyze this circuit, treat it as two separate problems. The problem to solve here is to find V1 when

$$V_0 = 10 + 11 \sin(377t)$$

Treat this as two separate problems

- A DC input (V0 = 10), and
- An AC input (V0 = 11 sin(377t)

Start with the DC input.

DC Analysis: Start with assuming

$$V_0 = 10$$

This is a phasor problem with $\omega = 0$. This means that inductors and capacitors become

$$L \to j\omega L = 0$$

$$C \to \frac{1}{j\omega C} = \infty$$

This results in the circuit being as shown in Figure 6

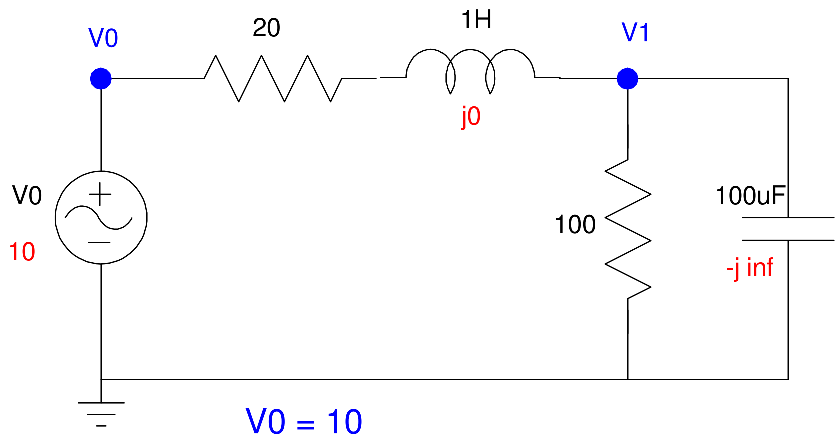

Figure 6: Circuit model when V0 = 10 (DC circuit)

The voltage node equations are then

$$V_0 = 10$$

$$\left(\frac{V_1 - V_0}{20}\right) + \left(\frac{V_1}{100}\right) + \left(\frac{V_1}{-j\infty}\right) = 0$$

which results in

$$V_1 = 8.333V$$

This is the DC term for V1. Note that voltage division also works:

$$V_1 = \left(\frac{100}{100+20}\right) \cdot 10$$

$$V_1 = 8.333V$$

AC Analysis: Next, assume V0 is the AC term

$$V_0(t) = 11\sin(377t)$$

This is a phasor problem where

$$\omega = 377$$

$$V_0 = 0 - j11$$

$$L \rightarrow j\omega L = j377\Omega$$

$$C \rightarrow \frac{1}{j\omega C} = -j26.5\Omega$$

which gives the circuit shown in Figure 7.

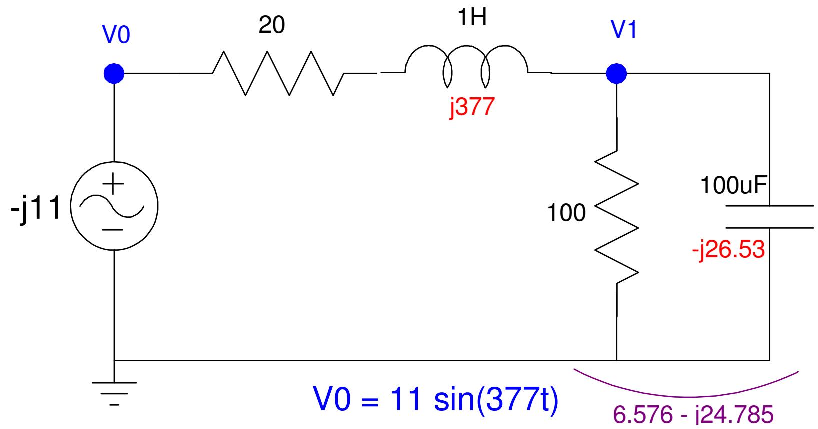

Figure 7: AC model for this circuit

The voltage node equations are

$$V_0 = -j11$$

$$\left(\frac{V_1 - V_0}{20 + j377}\right) + \left(\frac{V_1}{100}\right) + \left(\frac{V_1}{-j26.53}\right) = 0$$

Voltage division also works. The capacitor and resistor combine as

$$Z_1 = (100)||(-j26.53)$$

$$Z_1 = 6.576 - j24.785$$

By voltage division

$$V_1 = \left(\frac{(6.576 - j24.785)}{(6.576 - j24.785) + (20 + j377)}\right) \cdot (0 - j11)$$

$$V_1 = -0.241 + j0.763$$

$$V_1 = 0.800\angle 107.5^0$$

Note that the magnitude of V1 matches the result from CircuitLab (Vp = 795.5mV).

The total answer is the DC term plus the AC term

DC:

$$V_1 = 8.333$$

$$v_1(t) = 8.333$$

AC:

$$V_1 = -0.241 + j0.763$$

$$v_1(t) = -0.241\cos(377t) - 0.763\sin(377t)$$

Add these to get the total answer

$$\boxed{v_1(t) = 8.333 - 0.241\cos(377t) - 0.763\sin(377t)}$$

Note that this matches the results from CircuitLab. CircuitLab gave

- The DC term is 8.332V (the average of Vmax and Vmin)
- The AC term is 1.591Vpp (the difference: Vmax - Vmin)

Calculations give

- The DC term is calculated at 8.333V (matches)
- The AC term is 800mV peak (1.600Vpp)

Case 2: AC + C

Next, consider the case when V0 contains two AC terms as shown in Figure 8.

$$V_0 = 10\sin(300t) + 4\cos(500t)$$

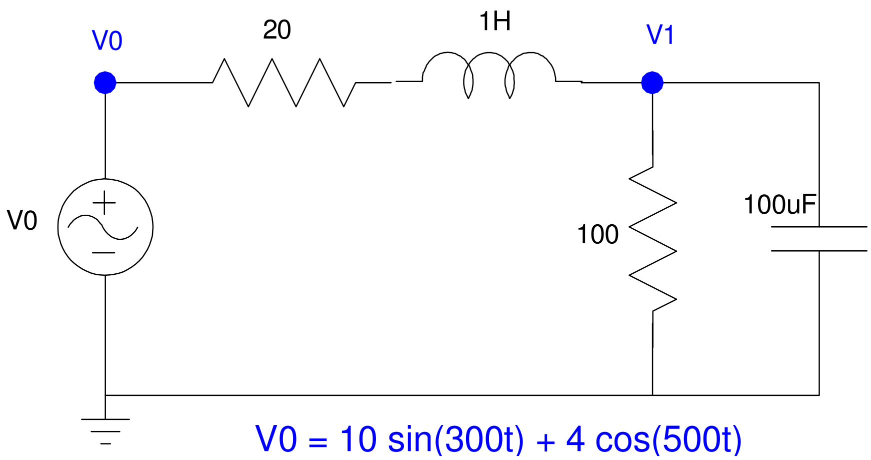

Figure 8: Find V1 when V0 contains two AC inputs at different frequencies

In CircuitLab, V1 can be found by applying two voltage sources in series as shown in Figure 9

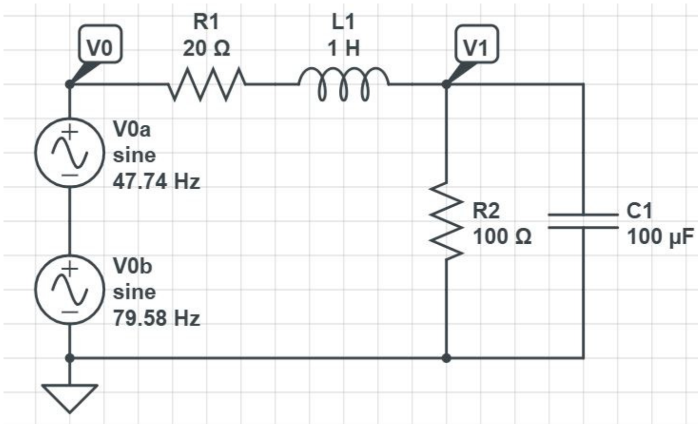

Figure 9: Two separate frequencies modeled as two AC voltage sources in series

Running a time-domain simulation produces a somewhat erratic output as shown in Figure 10. It looks odd, but that's what it is:

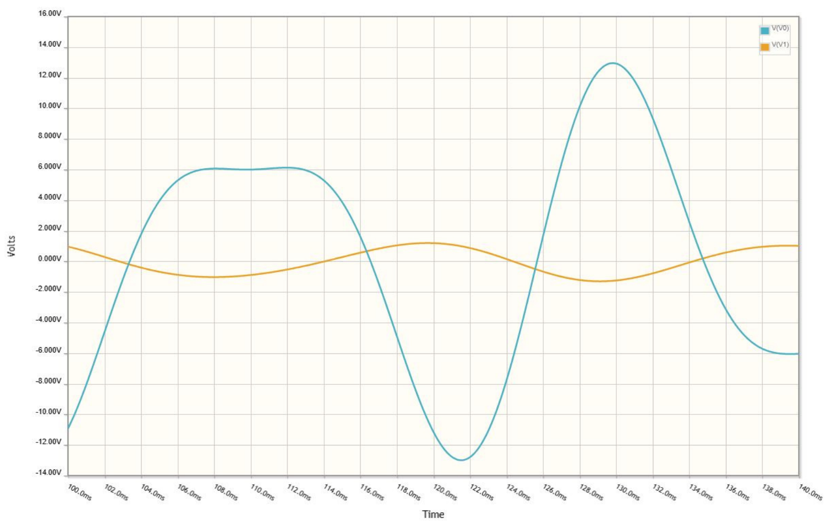

Figure 10: Voltages at V1 (blue) and V2 (orange) as found from CircuitLab

To find the equation for V1(t), treat this as two separate problems:

- V0 = 10 sin(300t)
- V0 = 4 cos(500t)

V1(t) will be the sum of these two separate solutions.

V1 = 10 sin(300t):

First, assume V1(t) is the 300 rad/sec input. This becomes a phasor problem with

$$\omega = 300$$

$$V_0 = 0 - j10$$

$$L \rightarrow j\omega L = j300$$

$$C \rightarrow \frac{1}{j\omega C} = -j33.33$$

which results in the circuit of Figure 11.

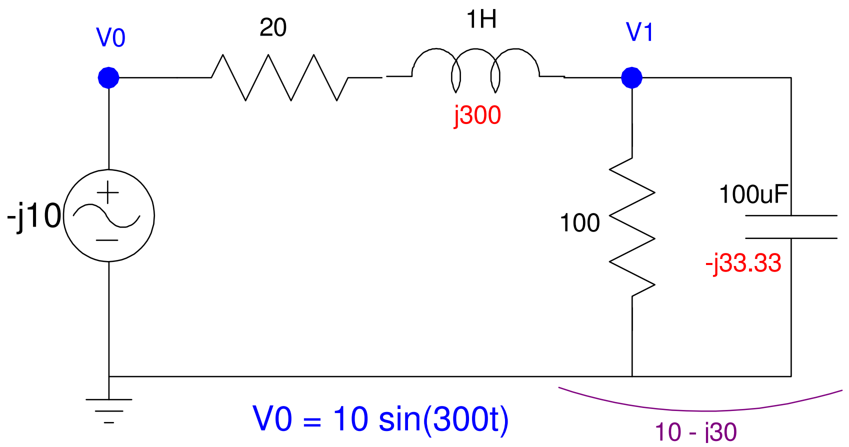

Figure 11: Circuit for the 300 rad/sec input

The voltage node equations then become

$$V_0 = -j10$$

$$\left(\frac{V_1 - V_0}{20 + j300}\right) + \left(\frac{V_1}{100}\right) + \left(\frac{V_1}{-j33.33}\right) = 0$$

Voltage division can also be used. The resistor and capacitor in parallel become

$$(100) || (-j33.33) = 10 - j30$$

By voltage division

$$V_1 = \left(\frac{(10 - j30)}{(10 - j30) + (20 + j300)}\right)(-j10)$$

$$V_1 = -0.4878 + j1.0568$$

which means

$$v_1(t) = -0.4878\cos(300t) - 1.0569\sin(300t)$$

V1 = 4 cos(500t)

Next, assume V1 is the 500 rad/sec input. This again becomes a phasor problem with

$$\omega = 500$$

$$V_0 = 4 - j0$$

$$L \rightarrow j\omega L = j500$$

$$C \rightarrow \frac{1}{j\omega C} = -j20$$

resulting in the circuit shown in Figure 12

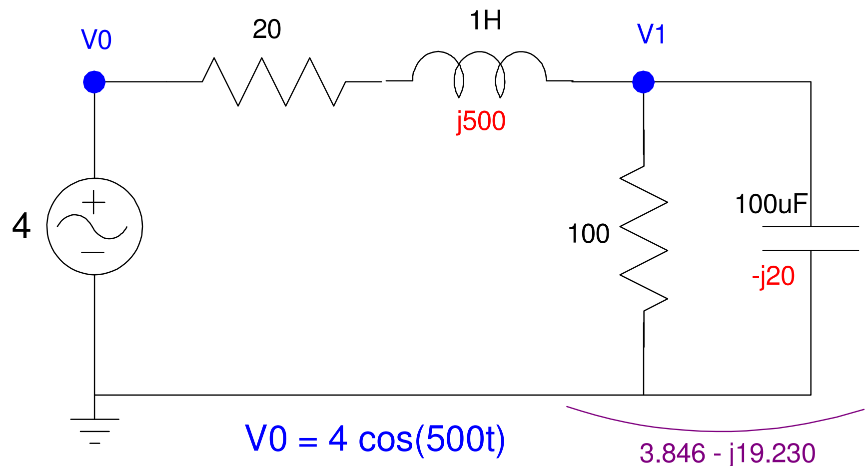

Figure 12: Equivalent circuit for the 500 rad/sec input

The voltage node equations are now

$$V_0 = 4$$

$$\left(\frac{V_1 - V_0}{20 + j500}\right) + \left(\frac{V_1}{100}\right) + \left(\frac{V_1}{-j20}\right) = 0$$

Voltage division can also be used. The resistor in parallel with the capacitor become

$$(100)||(-j20) = 3.846 - j19.230$$

By voltage division

$$V_1 = \left(\frac{(3.846 - j19.23)}{(3.846 - j19.23) + (20 + j300)}\right)(4)$$

$$V_1 = -0.1580 - j0.0398$$

meaning

$$v_1(t) = -0.1580\cos(500t) + 0.0398\sin(500t)$$

Total Answer: The total answer is the sum of the two outputs.

At 300 rad/sec

$$v_1(t) = -0.4878\cos(300t) - 1.0569\sin(300t)$$

At 500 rad/sec

$$v_1(t) = -0.1580\cos(500t) + 0.0398\sin(500t)$$

The total answer is thus

$$\boxed{v_1(t) = -0.4878\cos(300t) - 1.0569\sin(300t) - 0.1580\cos(500t) + 0.0398\sin(500t)}$$

This is the equation for the orange line in Figure 10.

Sidelight: Common mistake. A common mistake is to add the phasors together.

At 300 rad/sec

$$V_{1a} = -0.4878 + j1.0568$$

At 500 rad/sec

$$V_{1b} = -0.1580 - j0.0398$$

It's tempting to add these together to get the total answer.

$$V_1 \neq V_{1a} + V_{1b} = -0.6458 + j1.0170$$

This doesn't work. Each answer represents a sine wave at a different frequency. Sine waves at different frequencies don't add: they have to be represented with different terms. You likewise can't simplify the total answer given above

Summary

Superposition uses the property of linearity

$$f(x_1 + x_2 + x_3) = f(x_1) + f(x_2) + f(x_3)$$

This allows you to handle circuits where there are multiple sources at different frequencies.

If a circuit has an input containing two different frequencies, treat it as two separate problems

- Find the voltages and currents for each frequency separately
- Add the two together to get the total output