Thevenin Equivalents & Max Power Transfer

Background

Back when we were working with resistor circuits with DC inputs, we looked at Thevenin equivalents. A Thevenin equivalent is the simplest circuit which has the same load line as the original circuit. The procedure used to find the Thevenin equivalent was to

- Find the open-circuit voltage,
- Find the short-circuit current, and
- Find the impedance looking in when all independent sources are turned off.

The same works for AC circuits - only now you'll be working with phasors and complex numbers.

In this lecture, we'll be going over

- Finding the Thevenin equivalent for an RLC circuit with sinusoidal sources, and
- Finding the load that delivers maximum power to the load for this circuit.

Like previous lectures, we'll be using phasors representations for sources and RLC impedances as presented in Table 1.

	VI relationship	Phasor Notation
Voltage	$v(t) = a\cos(\omega t) + b\sin(\omega t)$	$V = a - jb$
Resistor	$v = iR$	$Z_R = R$
Inductor	$v = L\frac{di}{dt}$	$Z_L = j\omega L$
Capacitor	$i = C\frac{dv}{dt}$	$Z_C = \frac{1}{j\omega C}$

Table 1: Phasor representation for sources and RLC impedances

Finding the Thevenin Equivalent

It's probably easiest to explain this through an example. Consider the circuit of Figure 1. For this circuit, find

- The Thevenin equivalent
- The load, ZL, that maximizes the power to the load, and
- The maximum power that *can* be delivered to this load.

Note that the source, inductor, and capacitor have already been replaced with their phasor impedance. This implies that V0 is at a known, fixed frequency. If the frequency of V0 changes, this becomes a completely new problem.

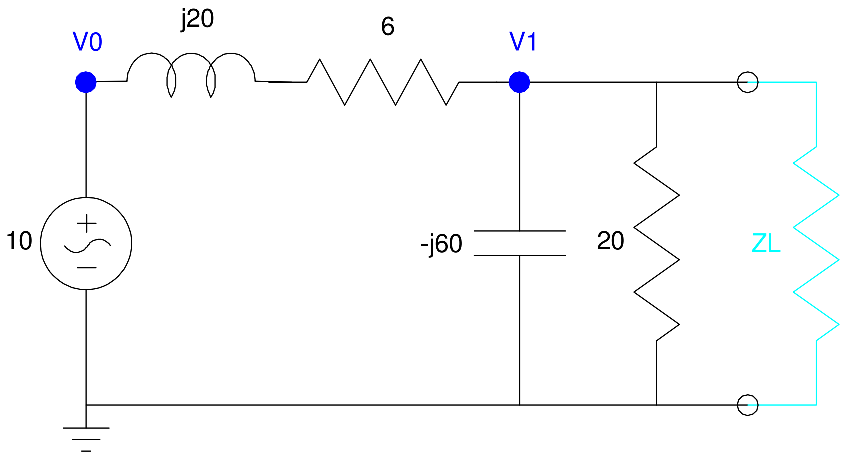


Figure 1: Find the Thevenin equivalent circuit as seen by ZL

Thevenin Equivalent: To find the Thevenin equivalent, two of the following three are needed:

- The open-circuit voltage (ZL = infinity)
- The short-circuit current (ZL = 0), and
- The equivalent resistance looking in when independent sources are turned off

Open-circuit voltage. Writing the voltage node equations give

$$V_0 = 10$$

$$\left(\frac{V_1 - V_0}{6+j20}\right) + \left(\frac{V_1}{-j60}\right) + \left(\frac{V_1}{20}\right) = 0$$

Solving, this gives

$$V_1 = 4.508 - j5.129$$

You could also get this using voltage division

$$-j60||20 = 18 - j6$$

$$V_1 = \left(\frac{(18-j6)}{(18-j6)+(6+j20)}\right)10$$

$$V_1 = 4.508 - j5.129$$

This is V(Thegenin).

Short-Circuit Current: To find this, set RL = 0 and compute the current flow through RL. By inspection

$$I_{sh} = \left(\frac{10}{6+j20}\right)$$

$$I_{sh} = 0.138 - j0.459$$

This is I(Norton)

Thevenin Impedance: This can be found a couple of ways. If you know V(open) and I(short), you know Rth:

$$Z_{th} = \left(\frac{4.5078 - j5.1295}{0.1376 - j0.4587}\right)$$

$$Z_{th} = 12.9637 + j5.9378$$

Another option is to turnoff the source and find the impedance looking in as shown in Figure 2. This gives

$$Z_{in} = (6 + j20)||(-j60)||(20)$$

$$Z_{in} = 12.9637 + j5.9378$$

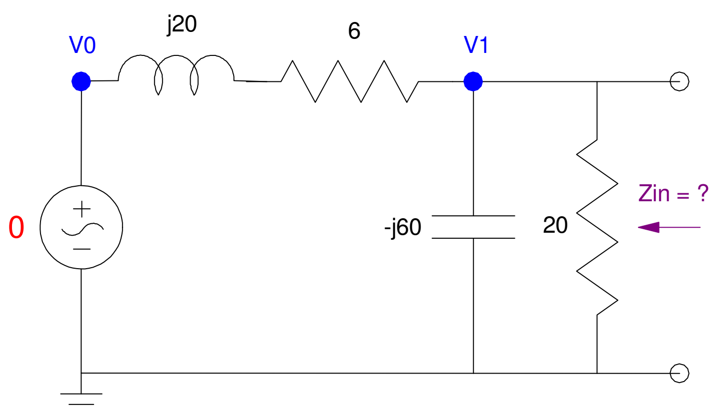


Figure 2: To find Rth, turn off the sources and find the impedance looking in

The Thevenin equivalent circuit for the circuit of Figure 1 is thus as shown in Figure 3.

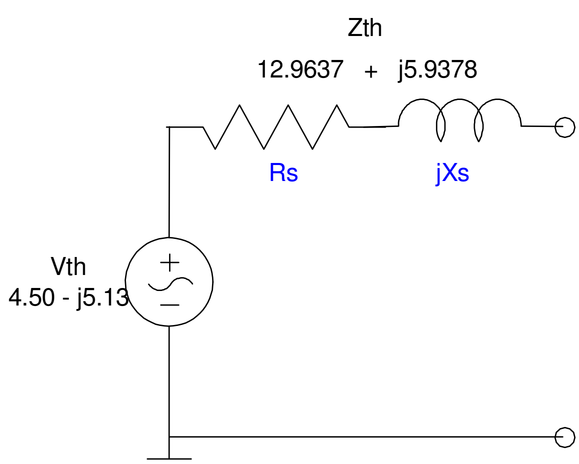


Figure 3: Thevenin equivalent for Figure 1

Note that the Thevenin voltage and Thevenin impedance are now complex numbers. The Thevenin impedance could be represented in series or parallel form.

- For Thevenin equivalents, series form is usually used.
- For Norton equivalents, parallel form is usually used.

$$I_n = I(short) = V_{th} \cdot Z_{th}$$

$$I_n = 0.138 - j0.459$$

$$Z_n = Z_{th} = 12.9637 + j5.9378$$

Usually, this Zn is expressed in parallel form

$$\frac{1}{Z_n} = \frac{1}{R_n} + \frac{1}{jX_n} = \frac{1}{12.9637+j5.9378}$$

This gives

$$R_n = 15.6834\Omega$$

$$jX_n = j34.2408\Omega$$

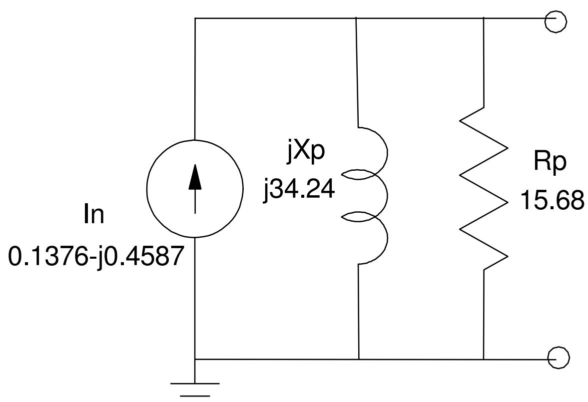


Figure 4: Norton Equivalent for Figure 1

Max Power Transfer

Switching gears. Let's go back to the circuit of Figure 1.

- What value of ZL maximizes the power to the load?
- What is the maximum power you can transfer to this load?

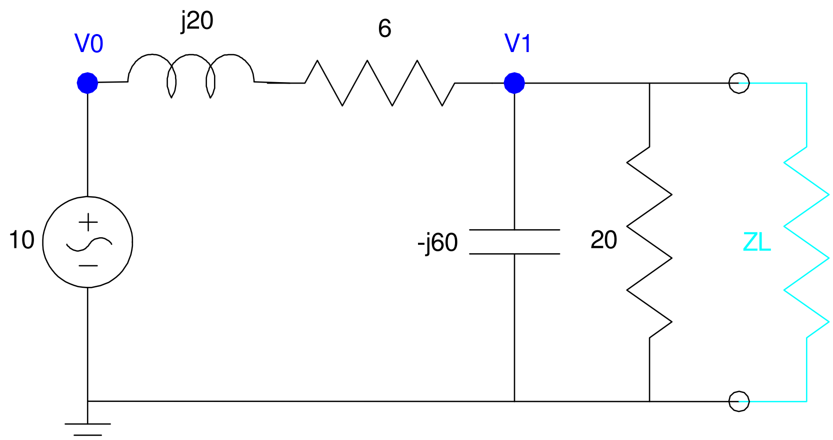


Figure 1 (repeat)

Before starting, note that power for complex numbers (phasors) is *slightly* different that for real numbers. With complex numbers, the complex conjugate is needed as show in Table 2. AC power also depends upon whether you are using peak voltages or RMS voltages.

DC	AC rms	AC peak
$P = V \cdot I$	$P = V_{rms} \cdot I_{rms}^*$	$P = \frac{1}{2} V_p \cdot I_p^*$
$P = I^2 R$	$P = \lvert I_{rms} \rvert^2 \cdot Z$	$P = \frac{1}{2} \cdot \lvert I_p \rvert Z$
$P = \frac{V^2}{R}$	$P = \frac{\lvert V_{rms} \rvert^2}{Z^*}$	$P = \frac{1}{2} \cdot \frac{\lvert V_{rms} \rvert^2}{Z^*}$

Table 2: DC and AC power

With AC signals, power has a real and a complex part.

- The real part of P is the work done (or heat produced),
- The complex part of P is the energy that bounces back and forth.

The maximum power to the load only concerns itself with the real part of the power: the part that actually does work.

One trick to finding the maximum power to the load is to use the Thevenin equivalent of the circuit. As far as the load is concerned, any circuit which produces the same load line looks the same as far as the load is concerned. Likewise, replace the circuit with its Thevenin equivalent as shown in Figure 5.

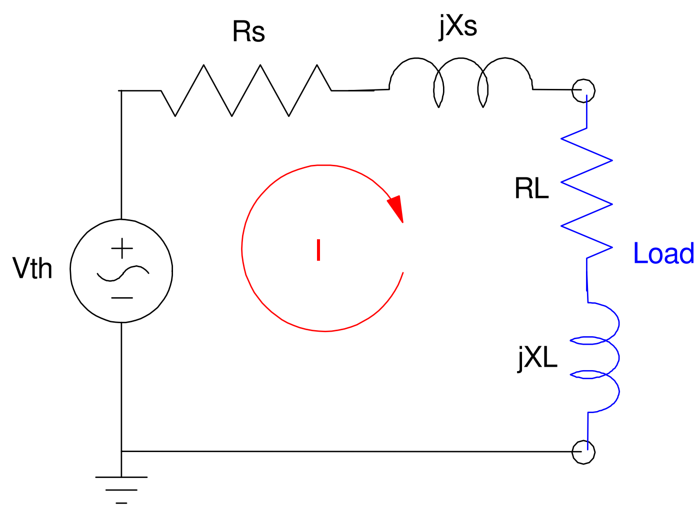


Figure 5: Replace the circuit with its Thevenin equivalent.

Note: Whether you use the Thevenin or Norton equivalent depends upon whether you use a series or parallel model for the load.

- If you're using a series model for the load, use the Thevenin equivalent
- If you're using a parallel model for the load, use the Norton equivalent.

In Figure 5, the load's RL and jXL are placed in series. Hence, the Thevenin equivalent is used.

Now, determining the voltage, current, and power to the load. Assuming rms units, the voltage at the load is

$$V_L = \left(\frac{R_L + jX_L}{(R_L + jX_L) + (R_s + jX_s)}\right) V_{th}$$

The loop current is

$$I_L = \left(\frac{V_{th}}{(R_L + jX_L) + (R_s + jX_s)}\right)$$

The power to the load is then

$$P = V_L \cdot I_L^*$$

$$P = \left(\frac{R_L + jX_L}{(R_L + R_s)^2 + (X_L + X_s)^2}\right) |V_{th}|^2$$

Taking the real part

$$real(P) = \left(\frac{R_L}{(R_L + R_s)^2 + (X_L + X_s)^2}\right) |V_{th}|^2$$

Find the maximum power with respect to jXL

$$\frac{dP}{dX_L} = 0 = \left(\frac{-R_L(2)(X_L + X_s)}{\left((R_L + R_s)^2 + (X_L + X_s)^2\right)^2}\right)$$

This happens when

$$X_L = -X_s$$

Now maximize power with respect to RL. Note that XL = -Xs

$$P = \left(\frac{R_L}{(R_L+R_s)^2+(X_L+X_s)^2}\right)|V_{th}|^2$$

$$P = \left(\frac{R_L}{(R_L+R_s)^2}\right)|V_{th}|^2$$

Maximizing

$$\frac{dP}{dR_L} = 0 = \left(\frac{(R_L+R_s)^2-R_L(2)(R_L+R_s)}{(R_L+R_s)^2}\right)|V_{th}|^2$$

$$(R_L+R_s)^2 - R_L(2)(R_L+R_s) = 0$$

$$(R_L+R_s)(R_s-R_L) = 0$$

This gives

$$R_L = R_s$$

The net results is

Make the load the complex conjugate of Zth (series load)

Note: if using a parallel model for the load, the result will be

Make the load the complex conjugate of Zn (parallel load)

The net result is

$$R_L + jX_L = R_s - jX_s$$

The maximum power is then

$$I = \frac{V_{th}}{R_s+R_L} = \frac{V_{th}}{2R_s}$$

$$P = V \cdot I^*$$

$$P = \frac{|V_{th}|^2}{2R_s}$$

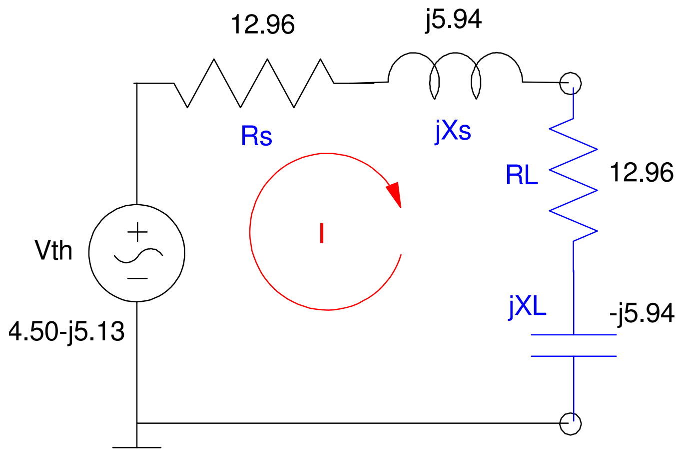


Figure 6: Max power to the load is when ZL is the complex conjugate of Zth (series load)

If you want to implement the load using a parallel model, the load is the complex conjugate of Zn as shown in Figure 7

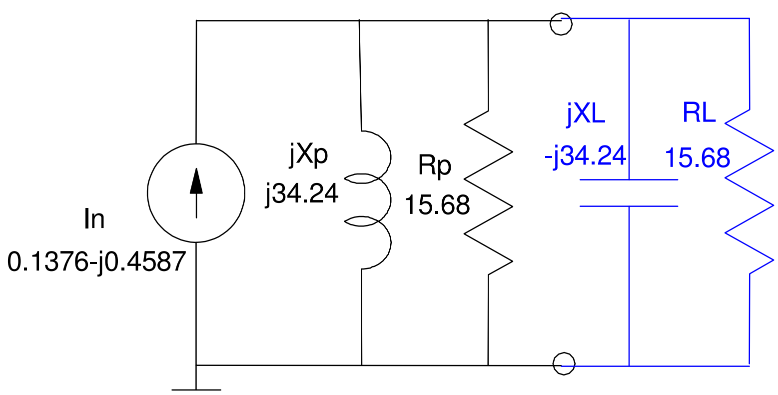


Figure 7: If using a parallel implementation for the load, make ZL the complex conjugate of Zn

"Capacitors Add Voltage"

If you look at power lines, often times you'll see capacitor banks on a telephone pole. These capacitors are used to help increase the voltage to customers at the end of a long line. You'll sometimes hear line workers say "capacitors add voltage".

The way this works is as follows. Often times, customers provide an inductive impedance looking back on the power line. This inductive load is due to the various motors in each household or farm. The net result is the Thevenin equivalent as seen by the last customer on a long transmission line looks inductive as shown in Figure 8.

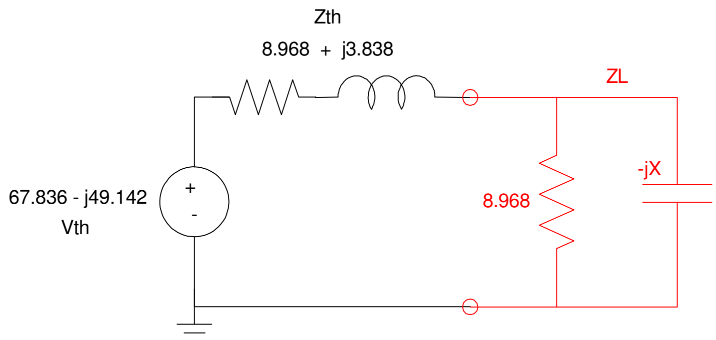


Figure 8: The last customer on a transmission line (shown in red) sees an inductive impedance looking in.

If you remove the capacitor, the inductance produces a voltage drop, lowering the voltage seen at the customer. If you add capacitance at the load, the capacitor's impedance cancels with the inductance, lowering the overall impedance. This increases the current to the load and raises the voltage at the load.

Only so much, however. If you add too much capacitance, the voltage at the load drops as shown in Figure 9. There's an optimal value of C: the value where the reactance of C cancels with the inductance of Zth.

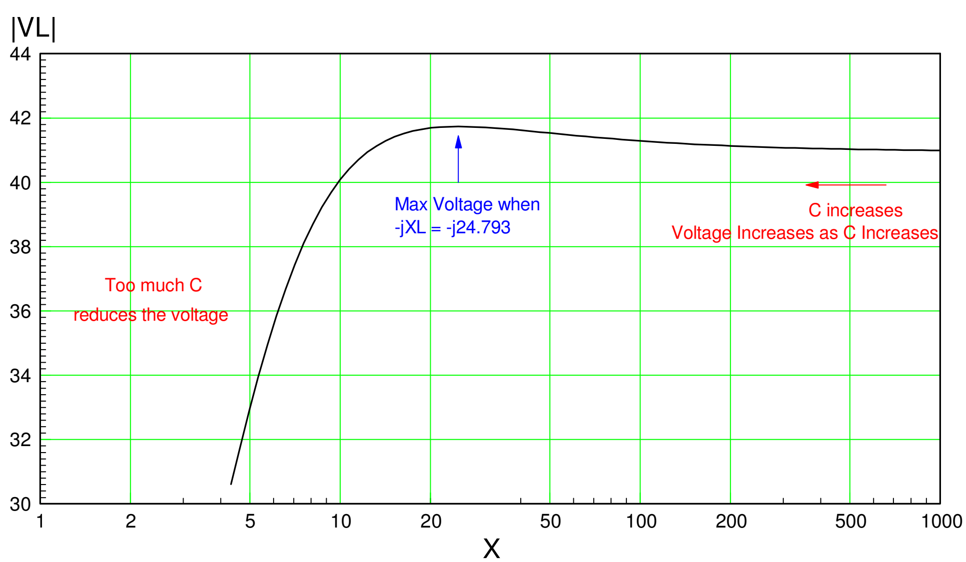
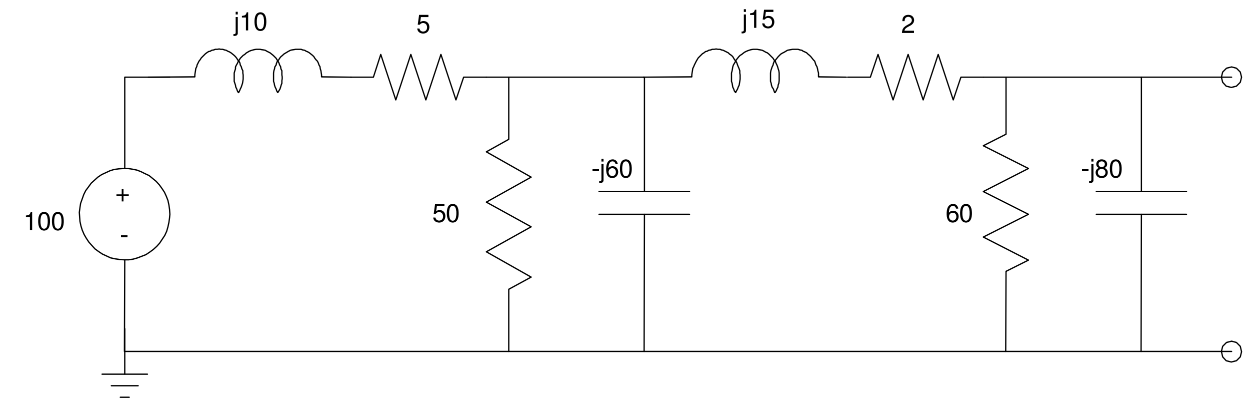


Figure 9: Voltage at the load as C varies. Starting with C=0 (infinite impedance), more C produces mode voltage. Up to a point.

Source Transformations

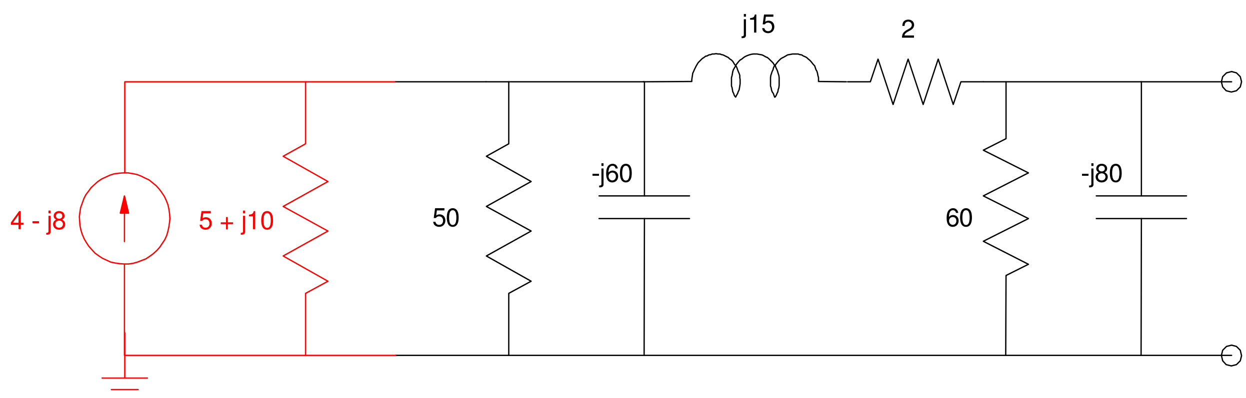
Source transformations also work with complex numbers. For example, determine the Thevenin equivalent for the following circuit:



Step 1: Convert to a Norton equivalent

$$Z_N = Z_{th} = 5 + j10$$

$$I_N = \frac{V_{th}}{Z_{th}} = 4 - j8$$



Combine impedances in parallel

$$(5 + j10)||(50)||(-j60) = 7.883 + j8.321$$

Convert to Thevenin

$$Z_{th} = Z_N = 7.883 + j8.321$$

$$V_{th} = I_N \cdot Z_N = (4 - j8) \cdot (7.883 + j8.321)$$

$$V_{th} = 98.102 - j29.781$$

Now find the Thevenin equivalent.

By voltage division:

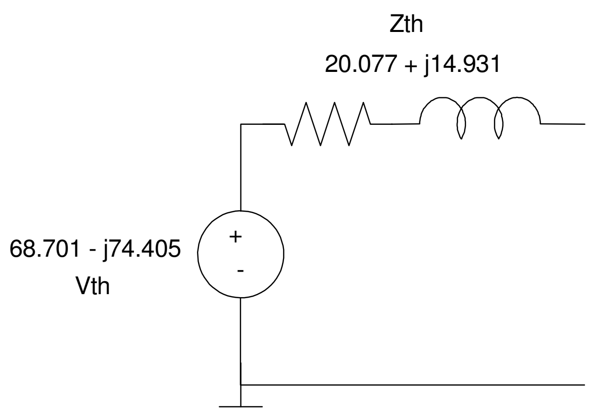
$$V_{th} = \left(\frac{(38.40 - j28.80)}{(38.40 - j28.80) + (7.883 - j29.781) + (2 + j15)} \right) (98.102 - j29.781)$$

$$V_{th} = 68.701 - j74.405$$

Zth: Turn off the source and measure the impedance

$$Z_{th} = ((7.83 - j29.781) + (2 + j15)) || (60) || (-j80)$$

$$Z_{th} = 20.077 + j14.931$$



Summary

Thevenin and Norton equivalents work with phasors just like they do with resistor circuits at DC. The only difference is you end up with a complex source and complex impedance.

Maximum power transfer happens when the load is the complex conjugate of the source's impedance:

- If the load is a series connection of RL + jXL, the load is the complex conjugate of the Thevenin impedance.
- If the load is a parallel combination or RL || jXL, the load is the complex conjugate of the Norton impedance.