Phasors and Voltage Nodes

Introduction

One of the tools that's useful in solving a given circuit is *Voltage Nodes*. The idea behind voltage nodes is to

- First, count the number of voltage nodes that are in a given circuit.
- Then, sum the currents to zero at each node to give N equations for N unknowns.

For example, in figure 1, at node V2 you would have

$$I_a + I_b + I_c = 0$$

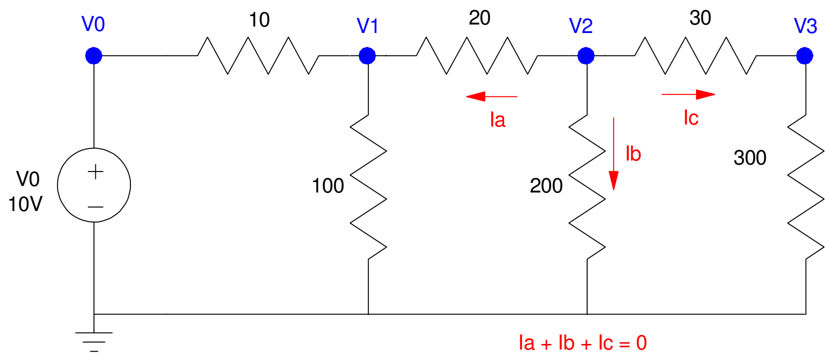

Figure 1: Voltage Nodes: Find the voltages so that currents sum to zero at each node.

Recall from previous lectures, the phasor representation for sources and impedances are as given in Table 1.

	VI relationship	Phasor Notation
Voltage	$v(t) = a\cos(\omega t) + b\sin(\omega t)$	$V = a - jb$
Current	$i(t) = a\cos(\omega t) + b\sin(\omega t)$	$I = a - jb$
Resistor	$v = iR$	$Z_R = R$
Inductor	$v = L\frac{di}{dt}$	$Z_L = j\omega L$
Capacitor	$i = C\frac{dv}{dt}$	$Z_C = \frac{1}{j\omega C}$

Table 1: Phasor representation for sources and impedances

Kirchoff's Voltage Nodes states that the current from a node must sum to zero. This applies at DC and at AC. The only difference is for AC circuits

- You are dealing with complex numbers (i.e. phasors), and
- Both the real and complex part of the current must sum to zero.

Essentially, if you don't mind using complex numbers, voltage nodes for AC circuits is just like AC analysis for DC circuits.

Example 1: RC Circuit

For example, use voltage nodes to determine the voltages and currents for the RC circuit in Figure 2.

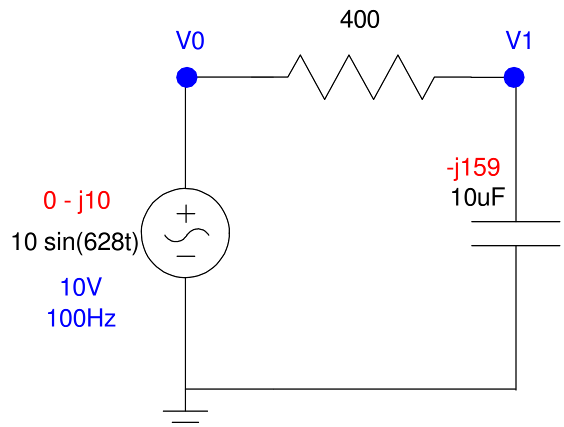

Figure 2: Single stage RC circuit: find the voltages and currents.

Step 1: Replace the voltage source and capacitor with their phasor value:

$$\omega = 628 \ \frac{rad}{\sec}$$

$$10\sin(628t) \to 0 - j10$$

$$10\mu F \to \frac{1}{j\omega C} = -j159\Omega$$

Step 2: Write the voltage node equations just like we did at DC:

$$V_0 = 0 - j10$$

$$\left(\frac{V_1 - V_0}{400}\right) + \left(\frac{V_1}{-j159}\right) = 0$$

Solve:

$$V_1 = \left(\frac{-j159}{-j159+400}\right)(0 - j10)$$

$$V_1 = 3.4326 - j1.3645$$

$$v_1(t) = 3.4326\cos(628t) + 1.3645\sin(628t)$$

This can be verified in CircuitLab. First, input the circuit using drag and drop

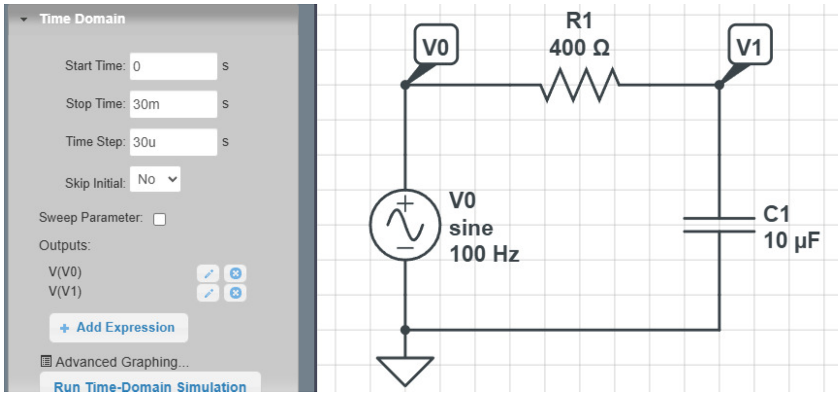

Figure 3: Input the circuit in CircuitLab.

Make the input a sine wave with

- no DC offset
- 10V amplitude
- 100Hz (628 rad/sec)

Run a transient response for

- 30ms (3 cycles)
- 30us step size (1000 points on the plot)

This result waveforms presented in Figure 4

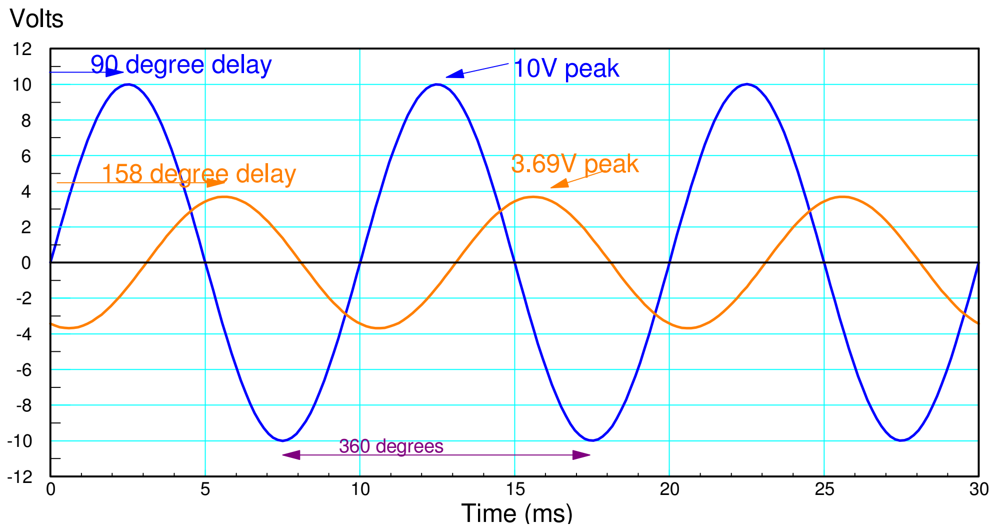

Figure 4: CircuitLab simulation result

Note that lab results are easiest to see in polar form:

- The peak voltage is the magnitude of the waveform
- The delay to peak is the phase shift

For V0 in Figure 4

$$V_0 = 0 - j10$$
$$= 10\angle - 90^0$$

Note that V0

- Has a peak of 10V (the magnitude), and
- The peak is delayed by 90 degrees (1/4th of a cycle)

For V1 (orange waveform)

$$V_1 = -3.43 - j1.36$$
$$= 3.69\angle - 158^0$$

From Figure 4

- V1 has a peak of 3.69V, and
- The peak of V1 is delayed by 158 degrees (43% of one cycle)

Example 2: 3-Stage RC Circuit

Next, consider a slightly more complicated RLC circuit as shown in Figure 5. Find the voltages assuming

$$V_0 = 7\cos(100t) + 3\sin(100t)$$

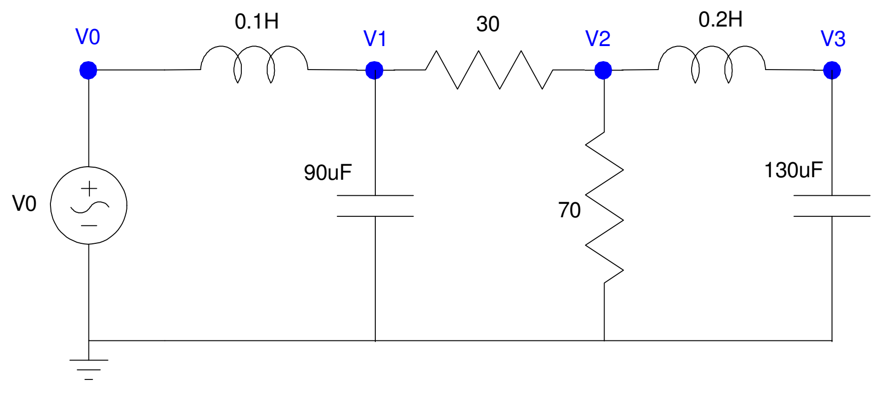

Figure 5: Use voltage nodes to find V0, V1, V2, and V3

The first step is to convert the sources and impedances into their phasor form.

$$V_0 = 7\cos(100t) + 3\sin(100t) \rightarrow 7 - j3$$

$$0.1H \rightarrow j\omega L = j10$$

$$0.2H \rightarrow j\omega L = j20$$

$$90\mu F \rightarrow \frac{1}{j\omega C} = -j111.1$$

$$130\mu F \rightarrow \frac{1}{j\omega C} = -j76.9$$

This gives the equivalent circuit of Figure 6

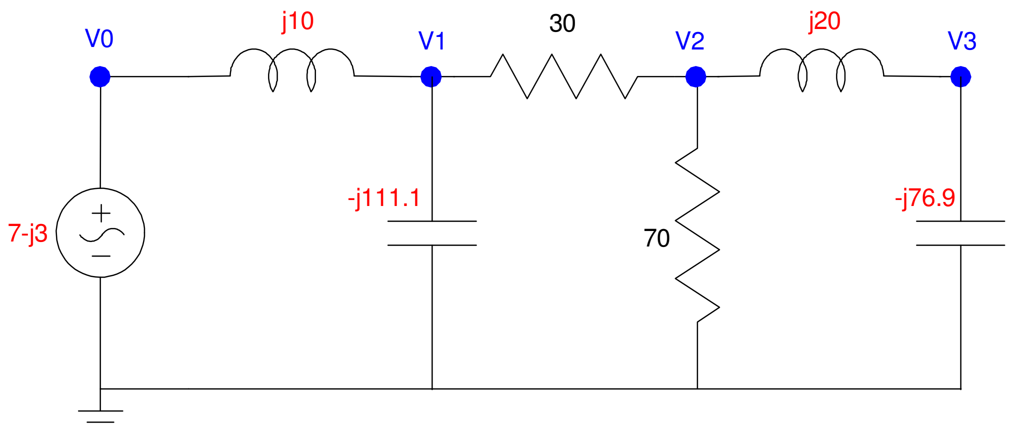

Figure 6: Phasor equivalent to Figure 5

Now write the voltage node equations. There are four voltage nodes. Hence, four equations are required for four unknowns. Start with the easy one (the voltage source)

$$V_0 = 7 - j3$$

Next, write the voltage node equations

$$\left(\frac{V_1 - V_0}{j10}\right) + \left(\frac{V_1}{-j111.1}\right) + \left(\frac{V_1 - V_2}{30}\right) = 0$$

$$\left(\frac{V_2 - V_1}{30}\right) + \left(\frac{V_2}{70}\right) + \left(\frac{V_2 - V_3}{j20}\right) = 0$$

$$\left(\frac{V_3 - V_2}{j20}\right) + \left(\frac{V_3}{-j76.9}\right) = 0$$

Now that you have four equations and four unknowns, solve. Group the terms

$$V_0 = 7 - j3$$

$$-\left(\tfrac{1}{j10}\right)V_0 + \left(\tfrac{1}{j10} + \tfrac{1}{-j111.1} + \tfrac{1}{30}\right)V_1 - \left(\tfrac{1}{30}\right)V_2 = 0$$

$$-\left(\tfrac{1}{30}\right)V_1 + \left(\tfrac{1}{30} + \tfrac{1}{70} + \tfrac{1}{j20}\right)V_2 - \left(\tfrac{1}{j20}\right)V_3 = 0$$

$$-\left(\tfrac{1}{j20}\right)V_2 + \left(\tfrac{1}{j20} + \tfrac{1}{-j76.9}\right)V_3 = 0$$

Place in matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \left(\frac{-1}{j10}\right) & \left(\frac{1}{j10} + \frac{1}{-j111.1} + \frac{1}{30}\right) & \left(\frac{-1}{30}\right) & 0 \\ 0 & \left(\frac{-1}{30}\right) & \left(\frac{1}{30} + \frac{1}{70} + \frac{1}{j20}\right) & \left(\frac{-1}{j20}\right) \\ 0 & 0 & \left(\frac{-1}{j20}\right) & \left(\frac{1}{j20} + \frac{1}{-j76.9}\right) \end{bmatrix} \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} (7-j3) \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Throw into Matlab and solve (this is where Matlab is *really* useful).

```
a1 = [1,0,0,0];
a2 = [-1/10j, 1/10j - 1/111j + 1/30, -1/30, 0];
a3 = [0, -1/30, 1/30 + 1/70 + 1/20j, -1/20j];
a4 = [0, 0, -1/20j, 1/20j - 1/76.9j];
A = [a1;a2;a3;a4];
B = [7-3j ; 0 ; 0 ; 0];
V = inv(A)*B

V0   7.0000 - 3.0000i
V1   7.6600 - 4.7718i
V2   3.6342 - 4.6815i
V3   4.9116 - 6.3271i
```

This tells you that

```
V0(t) = 7.00 cos(100t) + 3.00 sin(100t)
V1(t) = 7.65 cos(100t) + 4.77 sin(100t)
V2(t) = 3.63 cos(100t) + 4.68 sin(100t)
V3(t) = 4.91 cos(100t) + 6.32 sin(100t)
```

(real part is cosine, imag is -sine).   In Magnitude:

```
>> abs(V)

|V0|    7.6158
|V1|    9.0247
|V2|    5.9266
|V3|    8.0097
```

To verify these results, use CircuitLab. V0 is a 100 rad/sec sine wave. Converting to hertz (the units CircuitLab uses)

$$f = \frac{\omega}{2\pi} = 15.9155 Hz$$

V0(t) in polar form is

$$V_0 = 7 - j3$$

$$= 7.6158\angle -23.1986^0 \qquad \textit{cosine reference}$$

This phase shift assumes cosine waves. CircuitLab uses sine waves - which are cosine waves delayed another 90 degrees.

$$V_0 = 7.6158\angle -113.1986^0 \qquad \textit{sine reference (Circuitlab)}$$

Set the voltage source in CircuitLab to match V0

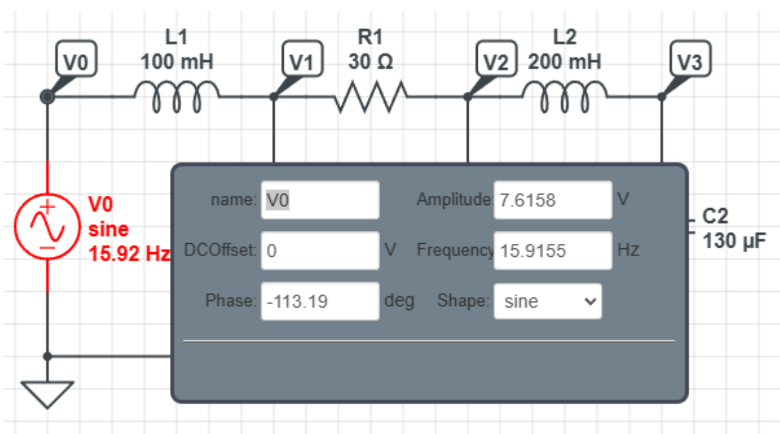

Figure 7: Match the amplitude, frequency, and phase to match V0

One cycle is 63ms (15.9Hz). Simulate for a few cycles (200ms) to get the steady-state response (phasors assume steady-state).

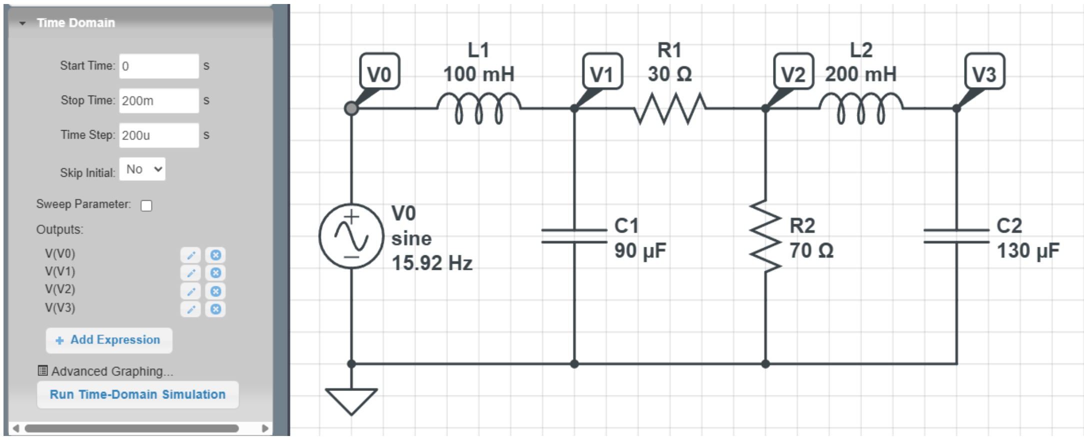

Figure 8: Simulate for several cycles to get the steady-state response

The resulting waveforms are presented in Figure 9.

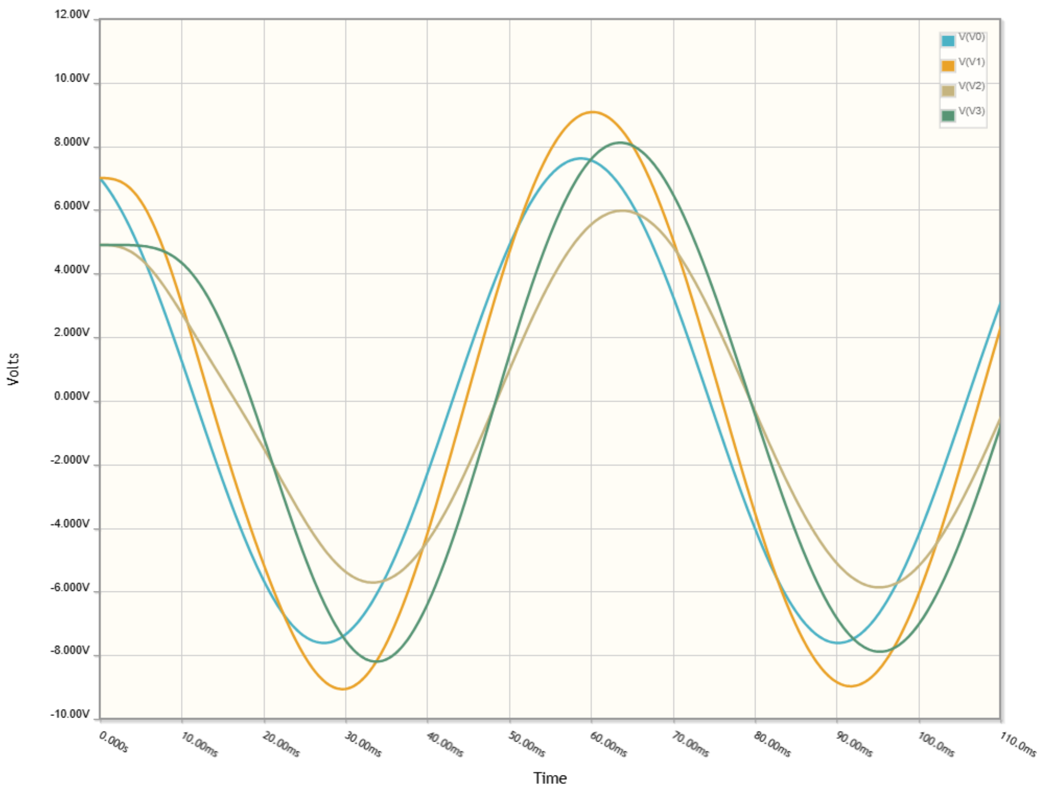

Figure 9: Resulting waveforms for V0, V1, V2, and V3

Note that the peak voltages from CircuitLab match up with calculations from Matlab as shown in Table 2.

Voltage	$\lvert V0 \rvert$	$\lvert V1 \rvert$	$\lvert V2 \rvert$	$\lvert V3 \rvert$
Calculated	7.6158	9.0247	5.9266	8.0097
CircuitLab	7.615V	8.995V	5.837V	7.936V

Table 2: Calculated vs. Simulated Voltages

Changing Frequencies

Here's another problem. Take the same circuit but change the frequency of V0. Now determine the voltages.

$$V_0 = 7\cos(200t) + 3\sin(200t)$$

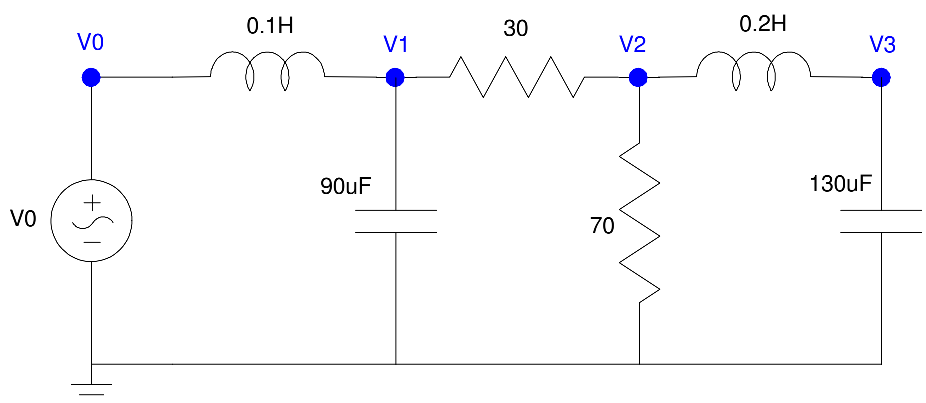

Figure 9: Find the voltages when the frequency of V0 changes.

If you change the frequency to 200 rad/sec, the phasor representation for V0 remains unchanged. It is still

$$V_0 = 7 - j3$$

The impedances of the inductors and capacitors *do* change, however, since they depend upon frequency

$$0.1H \rightarrow j\omega L = j20\Omega$$

$$0.2H \rightarrow j\omega L = j40\Omega$$

$$90\mu F \rightarrow \frac{1}{j\omega C} = -j55.55\Omega$$

$$130\mu F \rightarrow \frac{1}{j\omega C} = -j38.46\Omega$$

Essentially, if you change the frequency, it becomes a completely new problem. You have to start over and redo the voltage node equations.

Summary

Voltage nodes works the same for DC inputs and AC inputs. The only difference is with AC inputs, you're dealing with complex numbers (i.e. phasors).

When working with phasors, Matlab is *really* useful. Like before, when using voltage nodes, you end up solving N equations for N unknowns. Except with phasors, the equations contain complex numbers and the results are also complex numbers. That isn't a problem for Matlab. This is where Matlab really shines.