Mutual Inductance

Introduction

In a previous lectures, inductors were introduced. With inductors, one inductor can have an effect on other inductors. This is termed *mutual inductance*. Sometimes, mutual inductance is bad: it created cross-talk in a circuit. Sometimes, mutual inductance is good: it's how transformers and motors work.

In this lecture, we'll look at:

- Mutual inductance along with how to model it.
- Transformers: an electronic device which uses mutual inductance to change AC voltage levels
- Modeling a transformer using phasors from our last lecture, and
- Using transformers to transmit power over long distances.

The last item is important. Mutual inductance and transformers are how our power grid works. With them, utilities are able to transmit large amounts of energy thousands of kilometers at more than 90% efficiency.

Figure 1: Using transformers, utilities can transmit power thousands of kilometers

Capacitors

Capacitors are different from inductors. Unlike inductors, capacitors do *not* affect each other. The reason for this is capacitors store their energy in electrical fields whereas inductors store their energy in magnetic fields.

Since electrical charges *do* exist, electric field lines have a starting point and an ending point. For example, if you have a + and a - change, the field lines would look something like that shown in Figure 2.

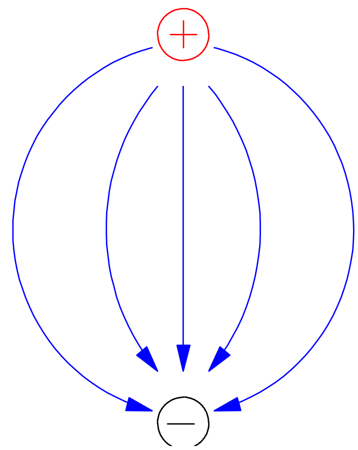

Figure 2: The field lines flow from a + charge to a - change

In a parallel plate capacitor, some of the field lines will extend beyond the plates of the capacitor. If these field lines overlap with another capacitor's field lines, there will be coupling. Since the spacing is so small, this coupling is effectively zero. This results in *mutual capacitance* (one capacitor affecting another) being a non-issue.

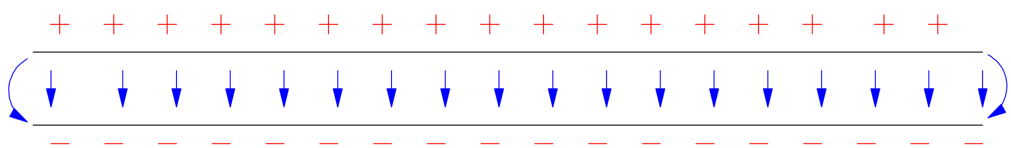

Figure 3: In a parallel plate capacitor, almost all of the field lines are confined between the plates of the capacitor

Inductors

Unlike capacitors, inductors store their energy in magnetic fields. Magnetic fields do not have a start or an end (magnetic monopoles do not exist). Instead, they form a closed path. For example, a coil with current flowing produces magnetic flux similar to the following:

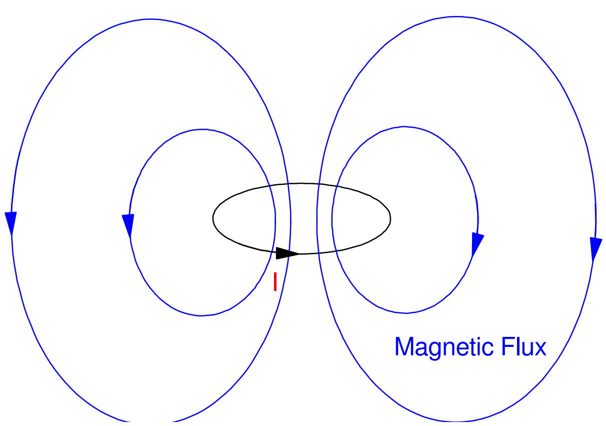

Figure 4:: Current traveling around a loop produces a magnetic field. The field lines do not have a beginning or end.

Since magnetic fields do not have a beginning or end, they cannot be easily contained. If the field lines of two inductors overlap, the two inductors affect each other. This is called *mutual inductance*. Sometimes this is good, sometimes this is bad.

Mutual Inductance

Mutual inductance is modeled as coupling between two inductors. Let 'k' be the percentage of the field lines shared by two inductors. When k=1, all of the magnetic flux is shared. An example of this would be a transformer where an iron core keeps most of the magnetic flux lines within the iron itself

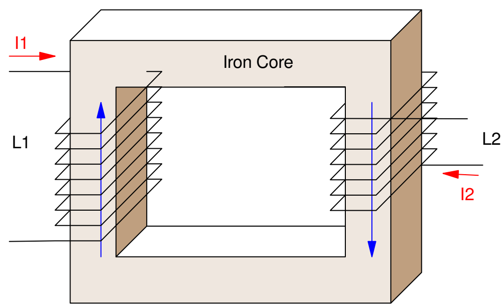

Figure 5: Example when the coupling coefficient (k) is one. Almost all of the flux lines from inductor L1 go through inductor L2

If two transformers are close to each other, some of the magnetic flux lines will be shared. In this case, $0<k<1$.

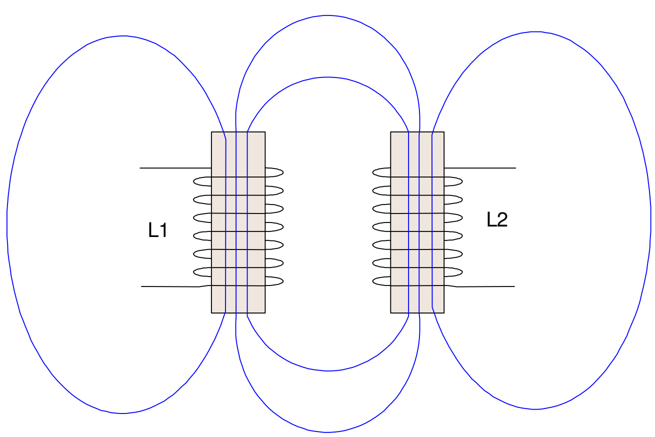

Figure 6: Example where $0 < k < 1$. Some of the magnetic flux lines are shared between two inductors.

As the inductors are moved further and further away, the coupling drops. Magnetic field strength drops as the cube of the distance, so coupling drops fairly quickly.

If two inductors are far apart or are perpendicular to each other, than the coupling is almost zero ($k = 0$).

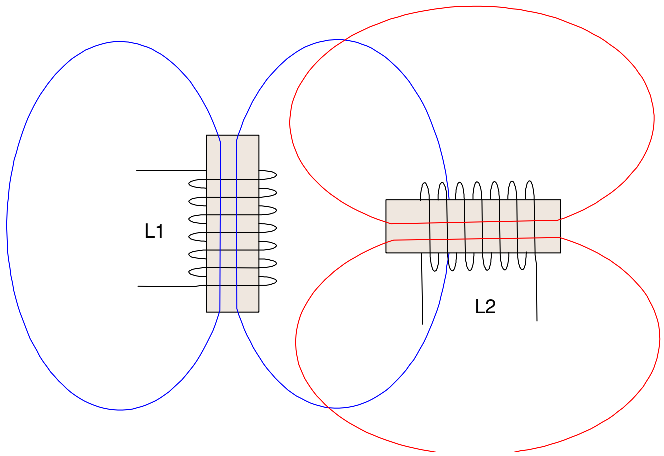

Figure 7: Example where k = 0. The inductors are perpendicular to each other, so they share no flux lines.

This makes circuit board design with inductors somewhat annoying: the placement of the inductors can have a significant influence on how the circuit behaves.

Electric Model for Mutual Inductance

If 'k' is the percentage of field lines that are common between two inductors, the mutual inductance, M, is

$$M = k\sqrt{L_1 L_2}$$

The T-circuit model for two couples inductors is then

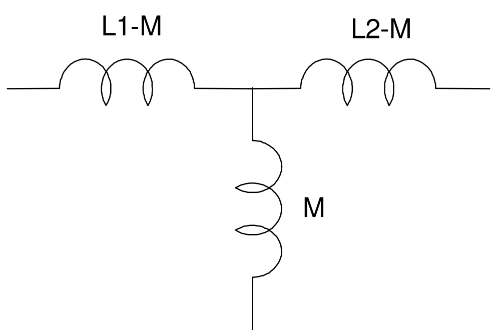

Figure 8: T-Circuit model for a circuit with two inductors and mutual inductance

Usage of Mutual Inductance: Transformers

Transformers are essentially two inductors which share a common iron core. The permeability of iron is much higher than air, resulting in most of the magnetic flux from one inductor going through the other inductor.

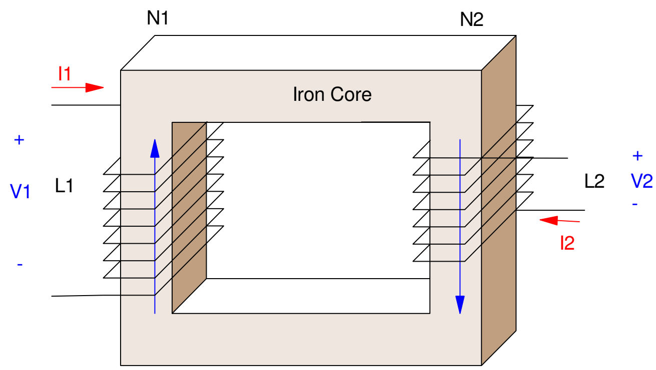

Figure 9: Transformer with N1 turns on inductor L1 and N2 turns on inductor L2

Since the magnetic field is the same for both inductors

$$\phi_1 = \phi_2$$

$$N_1 I_1 = N_2 I_2$$

$$I_2 = \left(\frac{N_1}{N_2}\right) I_1$$

Inductors are a way to increase or decrease current flow. Power has to balance

$$P_1 = P_2$$

$$V_1 I_1 = V_2 I_2$$

This results in

$$V_2 = \left(\frac{N_2}{N_1}\right) V_1$$

Transformers are a convenient way to change voltages and currents. This is why the power grid operates at 60Hz: transformers only work at AC (you need a changing flux to induce voltage and current).

Utility Transformers

Transformers form the backbone for electric utilities. With them, you can transform one voltage to another with ease. The reason to do this will be covered later in this lecture.

Transformers are also often times very large. Figure 10, for example, is a small transformer a utility would use in a residential neighborhood to provide power to maybe a dozen houses. This transformer

- Converts 13,200V to 240V (residential voltage)
- Weighs 470 pounds,
- Is 98.95% efficient (!), and
- Costs $3000 (ebay).

One of the main reasons transformers are so heavy is they rely on iron in their construction. Iron is about the only substance which supports strong magnetic fields. Hence, pretty much *all* transformers are made out of iron. A given volume of iron only supports a limited magnetic field. Hence, if you want to transmit a large amount of power (meaning use strong magnetic fields), you need a *lot* of iron. That's one of the main reasons transformers are so large and expensive.

Figure 10: Example of a 'small' utility scale transformer: 470 pounds and $3000.

Transformer Modeling

Transformers are inherently inductors: they store energy in their magnetic fields. Likewise, transformers are modeled as inductors.

Transformers also have losses. These losses are due to the copper windings:

- Copper has resistance. As current flows, heat (losses) is produced proportional to current squared.

The iron core also creates losses (heat):

- Iron has resistance. Current induced in the iron core produces I2R losses whenever the transformer is powered up.
- The iron core also has hysteresis losses. When powering a transformer with an AC voltage (60Hz for utilities), the constant flipping of the magnetic field in the iron creates heat (losses) whenever the transformer is powered up.

As a result, transformers are modeled as presented in Figure 11.

- The core is modeled as an inductance (jXc) along with a resistance (Rc). The core inductance is simply the inductance of the transformer on the primary side. The core resistance models all of the losses in the transformer when it's powered up without a load.
- The line models the secondary side. The line inductance models the inductance of the secondary side of the transformer. The line resistance models the total losses due to current flowing through the transformer (primary and secondary line losses lumped together).
- Connecting the two is an N1:N2 ideal transformer.

For the core and line impedances, either a parallel or series model could be used for each one. Tradition uses a parallel model for the core and a series model for the line.

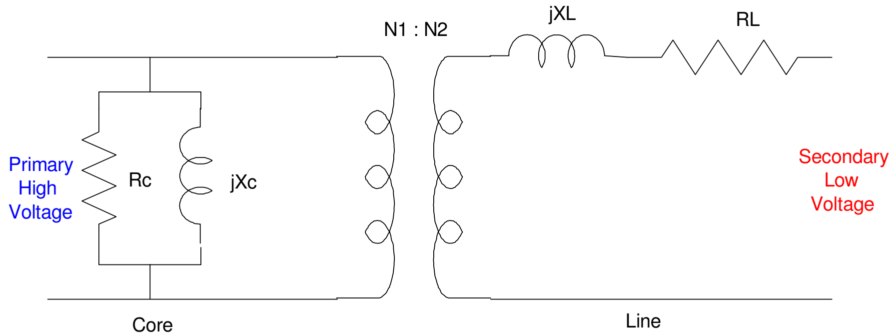

Figure 11: Model for a transformer

Transformer Testing

When a utility receives a batch of transformers, they are often tested to see if the actual transformers shipped meet the design specs. These teste consist of an open-circuit test and a short-circuit test.

Open-Circuit Test: The purpose of the open-circuit test is to determine the core model: Rc and jXc. To run this test,

- The secondary side is left open. This sets the line current to zero, making RL and jXL irrelevant.
- The primary side is excited with rated voltage.
- The voltage, current, and power used by the transformer is measured.

Then the core model can be computed: Rc and jXc. Note that it is important to use rated voltage with the open-circuit test. The iron losses vary with the applied voltage. If a lower voltage is used, you'll underestimated the core resistance, Rc.

The equations used to find Rc and jXc are

$$P = |V| \cdot |I| \cdot pf$$

$$Z_c = \left(\frac{|V|}{|A|}\right) \angle \arccos(pf)$$

$$\frac{1}{Z_c} = \frac{1}{R_c} + \frac{1}{jX_c}$$

Here, pf is termed the *power factor*. Power factor is the ratio of the actual power delivered to the load over the magnitude of Volts times Amps

$$pf = \left(\frac{W}{|V| \cdot |I|}\right)$$

If volts and amps are complex numbers, then power is a complex number

$$P = V \cdot I^*$$

and the power factor is the percentage of the power that's real (the losses) over the magnitude of the power

$$pf = \left(\frac{real(V \cdot I^*)}{|V \cdot I^*|}\right)$$

Utilities use power factor to keep track of how much of the current they deliver is used and paid for by the customer as a percentage of the total current they deliver to the customer. Ideally, the power factor should be 1.000 (meaning a purely resistive load and 100% of the current a customer uses goes on their bill). A purely inductive load has a power-factor of 0.000 (meaning the customer pays for 0% of the current used) - which upsets utilities. You'll get a fine if you have a power-factor of 0.000.

Short-Circuit Test: The second test is termed the *short-circuit test*. With this test,

- You short the primary side. This shorts out Rc and jXc so that they have no impact on the results.
- You apply a low voltage to the secondary side, and
- You measure the resulting voltage applied, current draw, the power consumed.

This lets you compute the line model: RL and jXL. Note that it is important to use *less* than rated voltage in this test. Transformers are not designed to be shorted out. If you use rated voltage in the short-circuit test, you'll burn out the transformer.

For example, assume you got the results of Table 2 when testing a transformer. From these results, determine the core model and the line model:

Test	Volts	Amps	Watts
Open-Circuit	Vp = 13,200V	Ip = 0.020A	250 Watts
Short-Circuit	Vs = 12V	Is = 50A	500W

Table 2: Open-circuit and short-circuit test results for a given transformer.

For the core model, use the open-circuit test. The calculations are

$$P = V \cdot I \cdot pf$$

$$pf = \left(\frac{250W}{13200V \cdot 0.020A}\right) = 0.947$$

$$Z_c = \left(\frac{V}{I}\right) \angle \arccos(pf)$$

$$Z_c = \left(\frac{13200V}{0.02A}\right) \angle \arccos(0.947)$$

$$Z_c = 660,000 \angle 18.737^0$$

$$\frac{1}{Z_c} = \frac{1}{R_c} + \frac{1}{jX_c}$$

$$R_c = 696.9k\Omega$$

$$jX_c = j2.054M\Omega$$

For the secondary side, use the short circuit test. Note that only 12V was used for this test. If 240V were used (20x more voltage), the current would be 20x more, or 1000A. That would fry the transformer.

$$P = V \cdot I \cdot pf$$

$$pf = \left(\frac{500W}{12V \cdot 500A}\right) = 0.0833$$

$$Z_L = \left(\frac{V}{I}\right) \angle \arccos(pf)$$

$$Z_L = \left(\frac{12V}{50A}\right) \angle \arccos(0.08333)$$

$$Z_L = 0.24 \angle 85.22^0$$

$$Z_L = R_L + jX_L$$

$$R_L = 0.020\Omega$$

$$jX_L = j0.239\Omega$$

The net result is a model for this transformer:

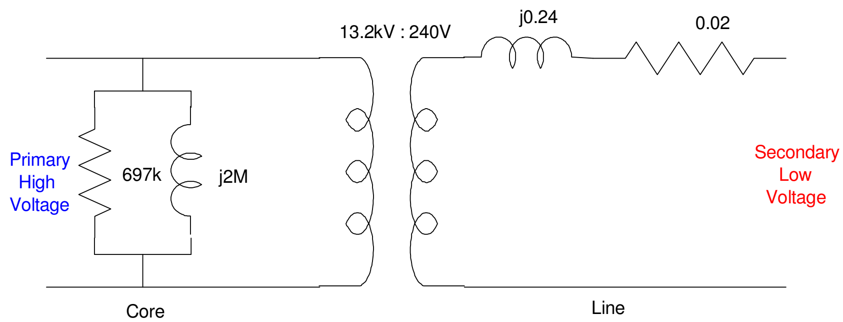

Figure 12: Experimental model for the transformer from the open-circuit and short-circuit test.

What you want for this model is

- Rc should be large: smaller Rc results in more losses any time the transformer is powered up.
- RL should be small: larger RL results in more losses when you use the transformer.

If either of these parameters are outside specs, the transformer would be returned as defective.

Transformer Uses: Power Grid

Transformers are the backbone of out power grid. With transformers, you can transmit power thousands of kilometers at more than 90% efficiency. Transformers only work for AC signals, however (transformers don't work at DC). This is the main reason our power grid operates at 60Hz (Tesla's design) rather than DC (Edison's design).

Figure 13: Electroboom has a good video explaining Edison's and Tesla's solution to build a power grid
https://youtu.be/S7C5sSde9e4?si=xloxxTr35nTrWEPd

For example, suppose you want to transmit power from western North Dakota to Minneapolis (1000km) as shown in Figure 14. Assuming copper wire, 1cm in diameter. The resistance of this wire would be

$$R = \frac{\rho L}{A}$$

$$R = \left(\frac{\left(1.68 \cdot 10^{-8}\ \Omega m\right)(1000km)}{\pi \cdot (0.005m)^2} \right)$$

$$R = 213\Omega$$

If you want to transmit 12kW at 120V (I = 100A), the line losses would be

$$P_{line} = I^2 R$$

$$P_{line} = (100A)^2 \cdot 213\Omega$$

$$P_{line} = 2130kW$$

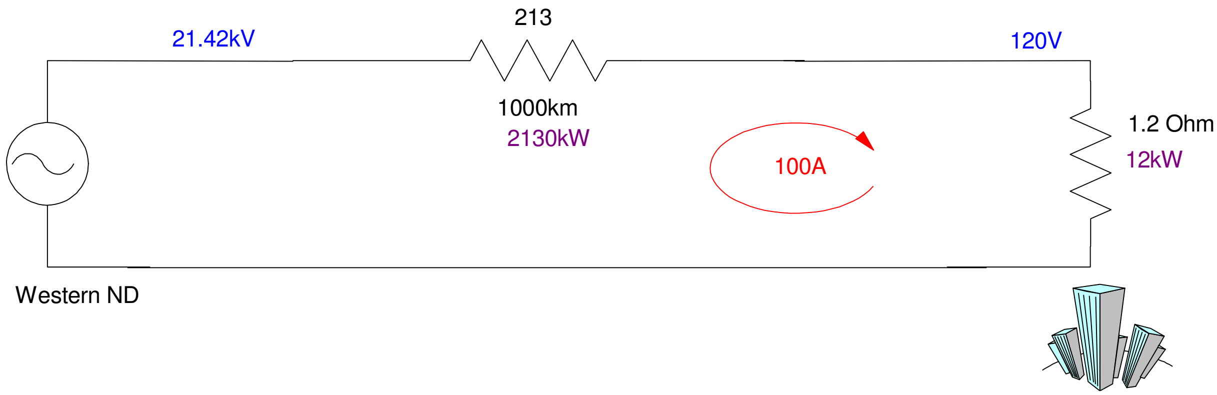

Figure 14: If you transmit power 1000km using DC, only 0.056% of the energy gets to the customer.

What this means is that if you operated the power grid at DC, like Edison proposed, in order to deliver 12kW to a load 1000km away, you would need to transmit 2142kW (2130kW + 12kW). In this case, only 0.056% of the energy transmitted gets to the load.

With Edison's model of a power grid, the farthest you could transmit power is about 5km. That means you would need to build generators every 5km or so.

In contrast, if you used an AC power grid, as Tesla proposed, you could use transformers to step up the voltage to 120kV and then step it back down to 120V as shown in Figure 15. Note that when you step up the voltage by 1000x, the power (P = VI) remains constant (conservation of energy). This means that the current drops 1000x.

1000 times less current results in 1-million times less losses.

$$I = 0.1A$$

$$P_{line} = (0.1A)^2 \cdot 213\Omega$$

$$P_{line} = 2.13W$$

In this case, 99.8% of the energy gets to the customer while only 0.2% is dissipated by the transmission lines.

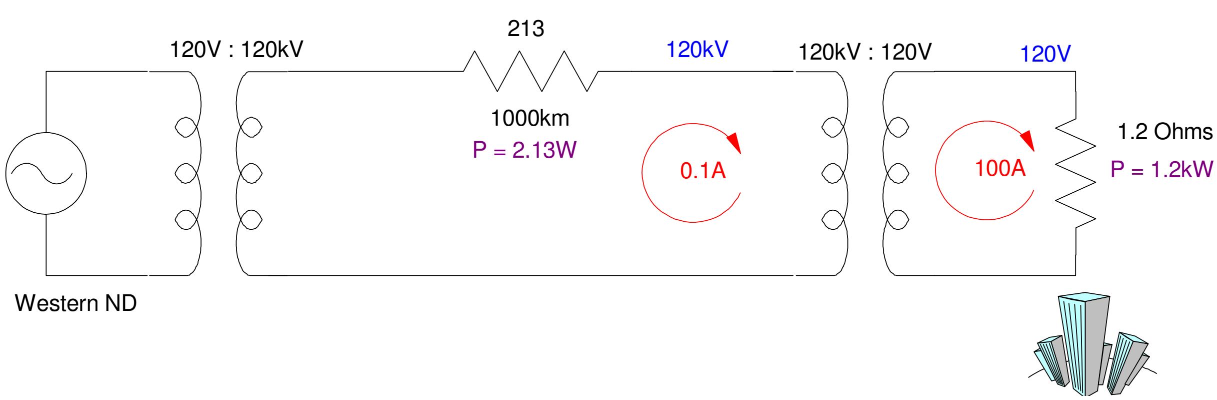

Figure 15: By transmitting power at 60Hz rather than DC, the efficiency of the transmission increases to 99.8%

Transformers and Impedance Matching

Transformers also change impedances. Suppose an impedance is connected to side 2

$$V_2 = R_2 I_2$$

Substituting, on side 1 you see

$$\left(\frac{N_2}{N_1}\right) V_1 = R_2 \left(\frac{N_1}{N_2}\right) I_1$$

$$V_1 = \left(\left(\frac{N_1}{N_2}\right)^2 R_2\right) I_1$$

The impedance seen through a transformer is changed by the square of the turns ratio.

This is used in some older stereos. Suppose the Thevenin resistance of your amplifier is 100 Ohms. If you drive an 8-Ohm speaker, only 7% of the energy gets to the speaker

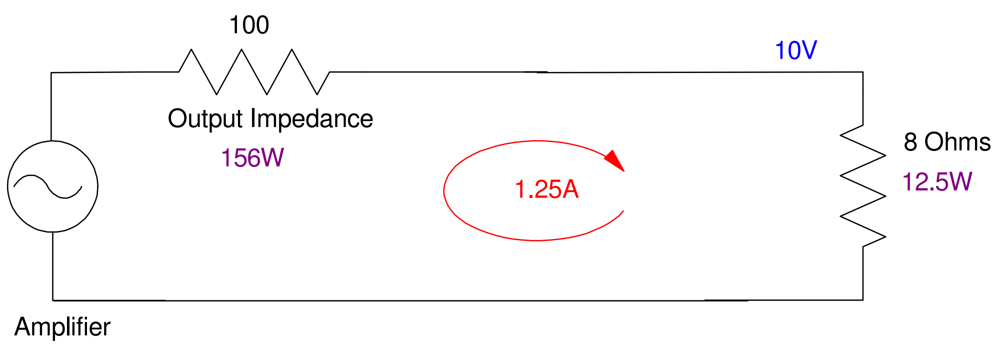

Figure 16: If an amplifier has an output impedance of 100 Ohms, only 7% of the energy gets to the speaker.

If you insert a 10:1 transformer before the speaker, the impedance as seen by the amplifier is

$$R = \left(\frac{10}{1}\right)^2 \cdot 8\Omega = 800\Omega.$$

Now, 89% of the energy gets to the speaker

$$eff = \eta = \frac{P_{load}}{P_{total}}$$

$$\eta = \left(\frac{I \cdot 800}{I \cdot (100+800)}\right) = \left(\frac{800}{900}\right) = 89\%$$

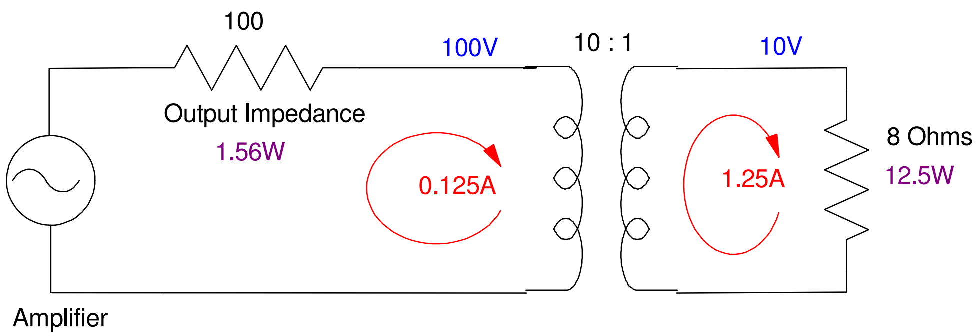

Figure 17: By inserting a transformer with a 10:1 turn ratio, the efficiency is increased to 89%. However, the voltage before the transformer needs to be 10x higher than the voltage at the speaker.

Note however that the voltage inside the amplifier is 10x higher than at the speaker. If you're driving the speaker at 100W

$$P = 100W = \frac{V^2}{8\Omega}$$

at the speaker you have 28V. At the amplifier side, the voltage will be 10x larger (280V).

Moral: If you see transformers in your stereo

- They're there to improve the efficiency of your stereo
- They also imply that the voltages inside the stereo can be very high.

Summary

Inductors store energy in magnetic fields. If two inductors are in a circuit and their magnetic fields overlap, you get *mutual inductance.*

Mutual inductance can be bad: it can create cross-talk between signals. Mutual inductance can also be good:

- Mutual inductance is how motors work
- Mutual inductance is now transformers work

Transformers are one of the most important electrical components. Transformers allow you to change voltages with ease. Transformers allow you to transmit large amounts of power over large distances. Transformers are likewise the backbone of our power grid.

Transformers don't work at DC, however. They only work for AC signals. This is the main reason our power grid operates at 60Hz rather than DC.