Phasors

Introduction

Back when we were working with DC signals, real numbers worked fine. With real numbers, the VI relationship for a resistor could be written as

V = IR

Once we start using sine waves and you add in inductors and capacitors, it *really* helps to use complex numbers for circuit analysis. With phasors, circuit analysis for AC signals becomes identical to circuit analysis for DC signals - only now with complex numbers. That's the goal: come up with a way to represent voltages, currents, and impedances for AC signals that works just like it did with DC circuit analysis. That way, everything we've learned so far still applies. Only now with complex numbers.

For example, in Figure 1, the phasor representation for a voltage source and impedances are shown in red. Circuit analysis can then commence using these voltages and impedances.

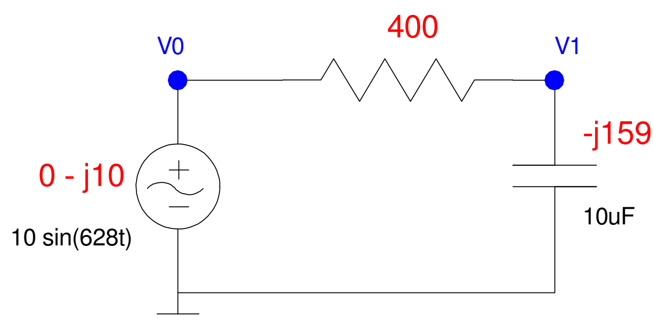

Figure 1: Phasor representation for an AC voltage source and RC impedances

This lecture coves

- Phasors,
- Representing voltages and currents using phasors,
- Representing RLC using phasors, and
- AC circuit analysis using phasors

Phasor Representation of Voltages

Euler's identity states

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

If you assume all functions are of the form of $e^{j\omega t}$, then

$$\cos(\omega t) = real(e^{j\omega t})$$

Likewise, the phasor (complex-number) representation for cosine is one

$$1 \leftrightarrow \cos(\omega t)$$

If you multiply by (a + jb)

$$(a+jb)e^{j\omega t} = (a+jb)(\cos(\omega t) + j\sin(\omega t))$$
$$= (a\cos(\omega t) - b\sin(\omega t)) + j(\cdots)$$

and take the real part you get the phasor representation for a generalized sine wave

$$a + jb \leftrightarrow a\cos(\omega t) - b\sin(\omega t)$$

For voltages and currents
- **The real part of a complex number represents the cos() term**
- **The imaginary part of a complex represents the -sin() term**

You can also represent voltages in polar form:

$$r\angle\theta \leftrightarrow r\cos(\omega t + \theta)$$

For example, determine the phasor representation of the voltage in Figure 1

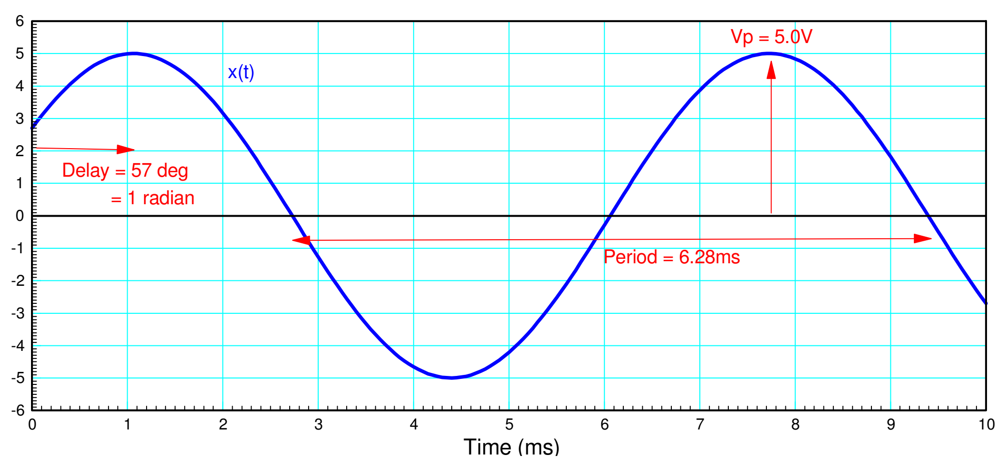

Figure 1: Determine the phasor representation for x(t)

Solution: The frequency comes from the period

$$T = 6.28ms$$

$$f = \frac{1}{T} = 159.2Hz = 159.2\frac{\text{cycles}}{\text{second}}$$

$$\omega = 2\pi f = 1000\frac{\text{rad}}{\text{sec}}$$

The peak voltage is the magnitude of x(t) in polar form

$$X = 5.0\angle\theta$$

The time delay of the peak determines the phase shift

$$\theta = -\left(\frac{\text{1ms delay}}{\text{6.28ms period}}\right)2\pi = -1.0\,\text{radian} = \text{-}57.3^0$$

meaning (in polar form)

$$x(t) = 5\cos(1000t - 57.3^0)$$

$$X = 5.0\angle - 57.3^0$$

or in rectangular form

$$X = 2.70 - j4.21$$

$$x(t) = 2.70\cos(1000t) + 4.21\sin(1000t)$$

The same holds for currents.

Phasor Representation for Impedance's

When dealing with AC signals, resistors, capacitors, and inductors can all be used. Phasor analysis converts each of these to a complex impedance.

Resistors: The VI relationship for a resistor is

$$V = IR$$

The phasor impedance of a resistor is R.

$$R \to R$$

Inductors: The VI relationship for an inductor is

$$V = L\frac{dI}{dt}$$

Assuming all signals are in the form of $e^{j\omega t}$, this means

$$V = L\frac{d}{dt}(e^{j\omega t}) = j\omega L\, e^{j\omega t} = j\omega L \cdot I$$

The impedance of an inductor is

$$L \to j\omega L$$

Capacitors: The VI relationship for a capacitor is

$$V = \frac{1}{C}\int I\, dt$$

Assuming all signals are in the form of $e^{j\omega t}$, this means

$$V = \frac{1}{C}\int (e^{j\omega t});dt = \left(\frac{1}{j\omega C}\right)e^{j\omega t} = \left(\frac{1}{j\omega C}\right)I$$

The impedance of a capacitor is

$$C \rightarrow \frac{1}{j\omega C}$$

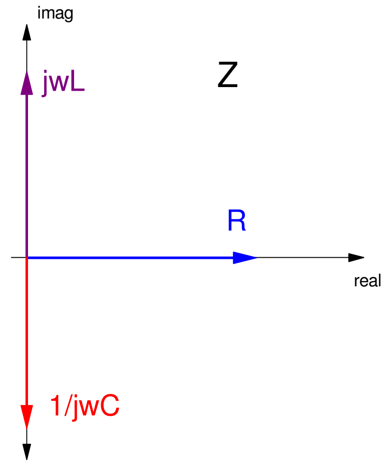

Figure 2: Complex impedance of resistors, inductors, and capacitors

ELI the ICE Man

- For inductors (L), voltage leads current (ELI)
- For capacitors (C), current leads voltage (ICE).

ELI: If $\omega L = 1$ then

$$L \rightarrow j\omega L = j = 1\angle 90^0$$

$$V = I \cdot j\omega L = I \cdot 1\angle 90^0$$

Voltage leads current (ELI) for inductors

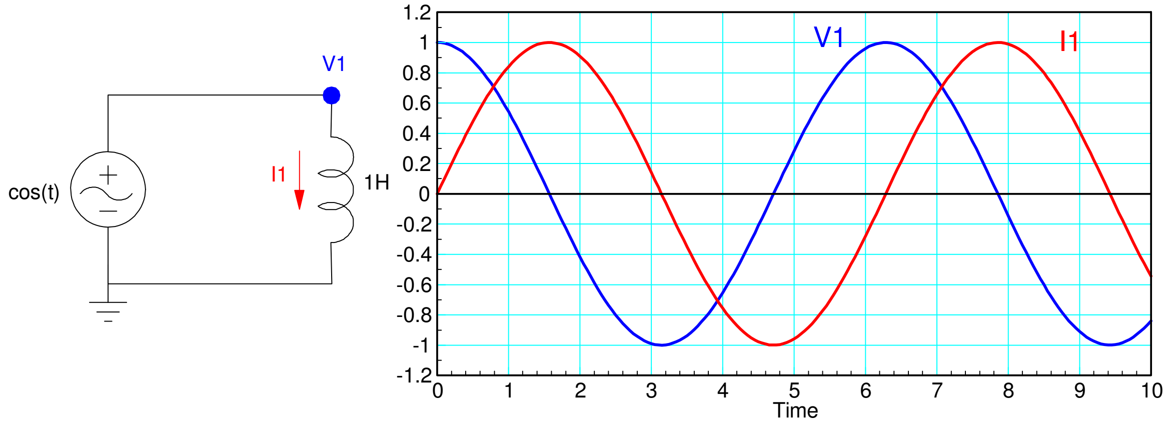

Figure 3: Voltage leads current for inductors (ELI)

ICE: For capacitors, if $\omega C = 1$, then

$$\frac{1}{j\omega C} = -j = 1\angle -90^0$$

$$V = I \cdot 1\angle -90^0$$

Voltage lags current by 90 degrees for capacitors (ICE)

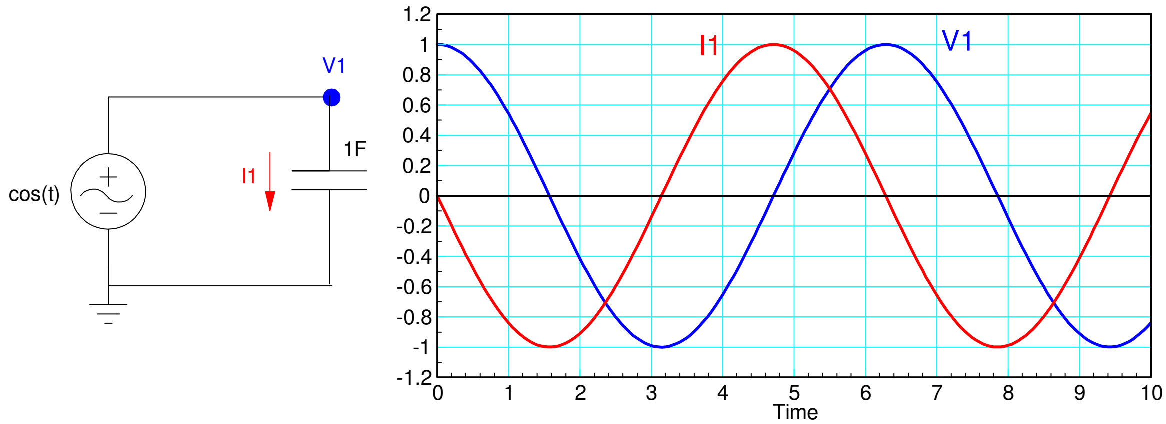

Figure 4: For capacitors, current leads voltage (ICE)

This is summarized in Table 1

Component	Phasor Representation
V = a cos(wt) - b sin(wt) I = a cos(wt) - b sin(wt)	a + jb
R	R
L	jwL
C	1 / jwC

Table 1: Phasor representation for voltages, currents, and impedances

Simplification of RLC circuits

The beauty of phasors is that everything we did for DC analysis also works for AC analysis. For DC analysis (resistor circuits),

- Resistors in series add
- Resistors in parallel add as $\left(\frac{1}{R_1} + \frac{1}{R_2} + ..\right)^{-1}$

For AC circuits, this still holds - only now using complex numbers

For example, determine the impedance from point a to b (Zab) in Figure 5:

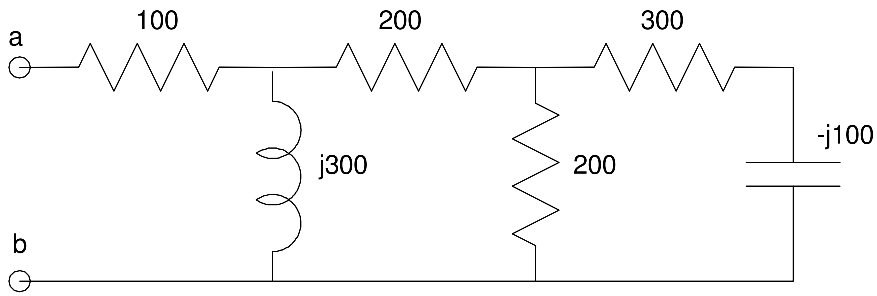

Figure 5: Find the impedance Zab

Solution:

-j100 and +300 are in series

-j100 + 300 = 300 - j100

This is in parallel with 200

$$0)||(300-j100) = \left(\frac{1}{200}+\frac{1}{300-j100}\right)^{-1} = 123.07-j15.38$$

Which is in series with 200

$$(200)+(123.07-j15.38) = 323.07-j15.38$$

Which is in parallel with +j300

$$(j300)||(323.08-j15.38) = 156.85+j161.83$$

which is in series with 100

$$(100)+(156.85+j161.83) = 256.85+j161.83$$

Answer:

$$Z_{ab} = 256.85+j161.83$$

Solving in Matlab

```
>> Z2 = 1 / (1/200 + 1/(300 - j*100) )

Z2 =  1.2308e+002 -1.5385e+001i

>> Z1 = 1 / (1/(j*300) + 1/(200 + Z2) )

Z1 =  1.5685e+002 +1.6183e+002i

>> Zab = 100 + Z1

Zab =  2.5685e+002 +1.6183e+002i
```

Solving with an HP42

```
300
enter
-100
complex
1/X
200
1/X
+
1/X     Z2 = 123.0769 - j15.3846
200
+
1/X
0
enter
300
complex
1/X
+
1/X     Z1 = 156.8465 + j161.8257
100
+       Zab = 256.8465 + j161.8257
```

Solving in CircuitLab

Zab tells you that current and voltage are related by

$$V = I \cdot Z_{ab}$$

$$V = I \cdot (256.8456 + j161.8257)$$

$$V = I \cdot (303.54\angle 32.21^0)$$

Translation:

- The voltage will be 303.54 times larger than the current
- Voltage leads current by 32.21 degrees

Let ω= 1 rad/sec

- $\omega = 2\pi f$
- f = 0.1591 Hz
- L = Z, C = 1/Z

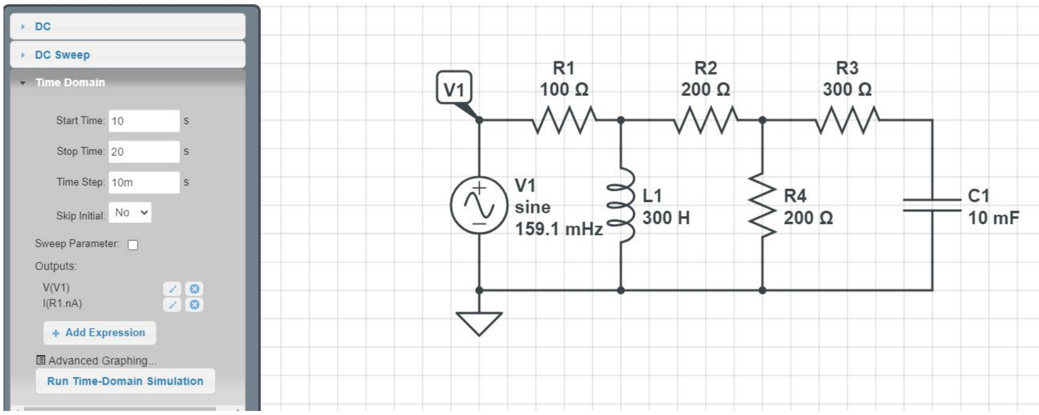

Figure 6: Add a test voltage (V1) and find the current draw to find the impedance

Plot voltage and current

- $Z_{ab} = 303.54\angle 32.21^0$
- The peak voltage is 303 times the peak current
- The peak voltage is 32.21 degrees before the peak current

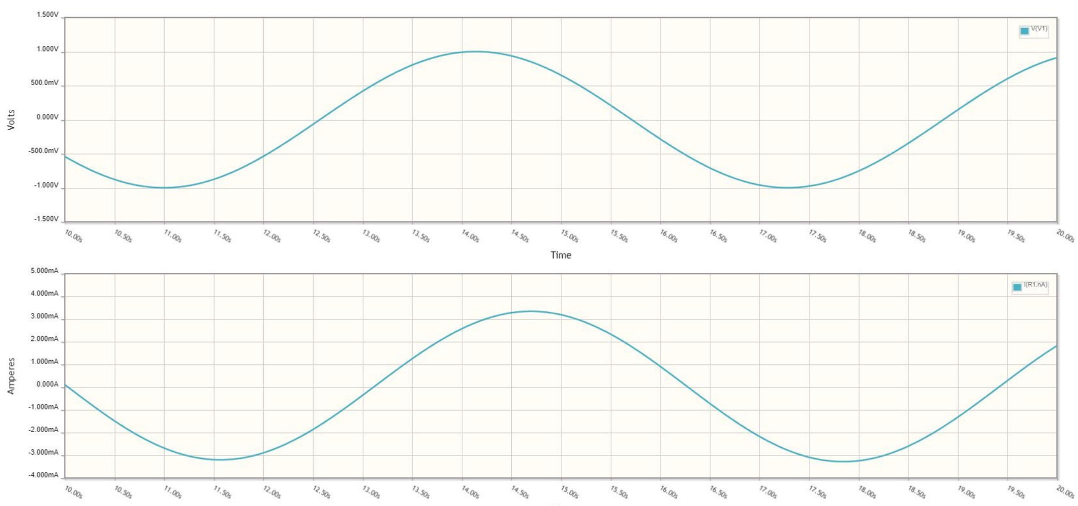

Figure 7: Voltage (top) and current (bottom). The ratio of the phasor representation of V/I gives Zab

Example 2: Determine Zab

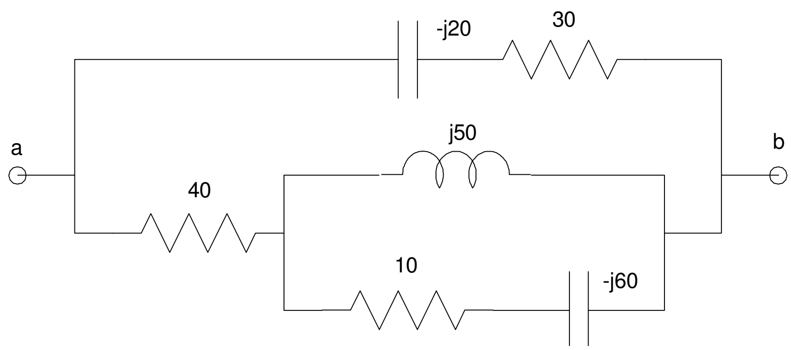

Figure 8: Simplify to find the impedance Zab

Solution

$$(10) + (-j60) = 10 - j60$$

$$(10 - j60)||(j50) = 125.00 + j175.00$$

$$(125.00 + j175.00) + (40) = 165.00 + j175.00$$

$$5.00)||(30 - j20) = 31.42 - j14.98$$

answer:

$$Z_{ab} = 31.42 - j14.98$$

Note that it *really* helps to have a calculator that can do complex operations. Matlab can do this. HP42s can do this.

In Matlab

```
>> Z1 = 10 - j*60

Z1 =  10.0000 -60.0000i

>> Z2 = 1 / (1/Z1 + 1/(j*50))

Z2 =  1.2500e+002 +1.7500e+002i

>> Z3 = Z2 + 40

Z3 =  1.6500e+002 +1.7500e+002i

>> Z4 = 1/( 1/Z3 + 1/(30-j*20) )

Z4 =  31.4263 -14.9799i
```

Solve using an HP42

```
10
enter
-60
complex
1/X
0
enter
50
complex
1/X
+
1/X           Z2 = 125 + j175
40
+             Z3 = 165 + j175
1/X
30
enter
-20
complex
1/X
+
1/X           Zab = 31.4263 - j14.9799
```

Solve with CircuitLab:

- Same trick as before: Let $\omega = 1$

$$V = I \cdot (31.4263 - j14.9799)$$

$$V = I \cdot (34.81\angle -25.49^0)$$

This means

- Voltage should be 34.81 times larger than current
- Voltage should lag behind current by 25.49 degrees.

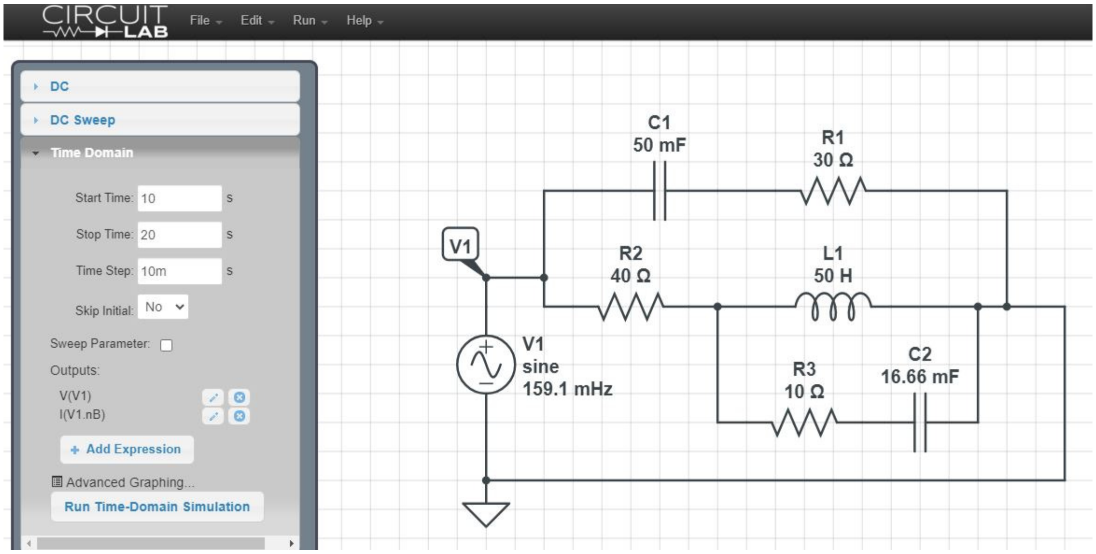

Figure 9: Apply a test voltage to find the impedance

After running a time-domain simulation, the voltages and currents are found as presented in Figure X:

- Voltage is 34.81 times the current
- Voltage lags current by 25.49 degrees
- $Z_{ab} = 34.81\angle -25.49^0$

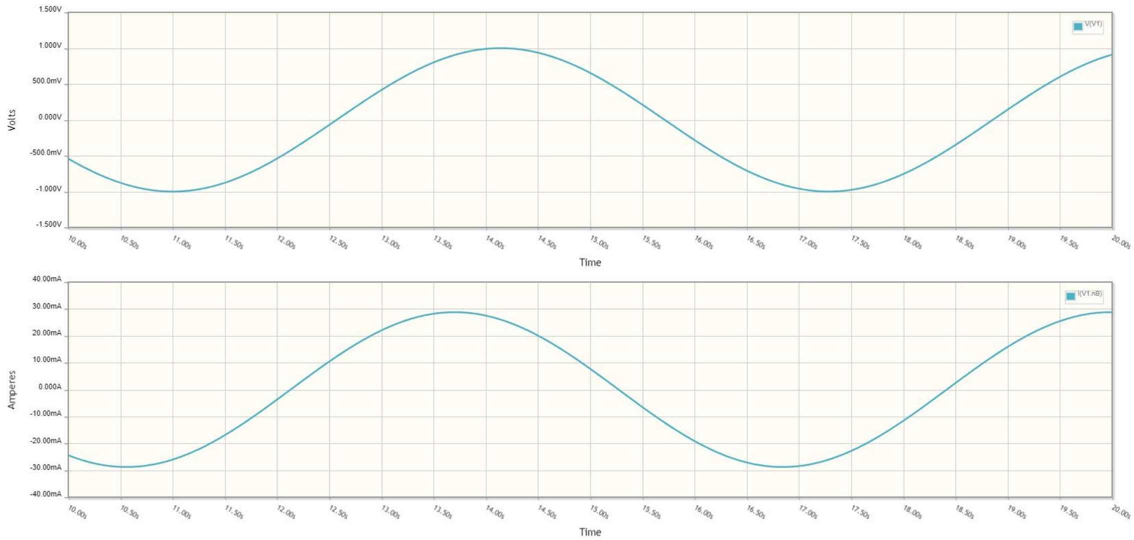

Figure 10: Voltage (top) and current (bottom) result in the ratio being Zab

Summary

Real numbers work well when analyzing DC circuits

Complex numbers work well when analyzing AC circuits

Everything that we did with DC circuits with AC circuits - only you wind up with complex numbers

For voltages

- The real part represents the cosine term
- The complex part is represents the minus-sine term

For impedances

- Resistors are real (R)
- Inductors are +jX (Z = jwL)
- Capacitors are -jX (Z = 1/(jwC))