Inductors

Capacitors deal with voltage and store energy in an electric field. Inductors, on the other hand, deal with current and store energy in a magnetic field.

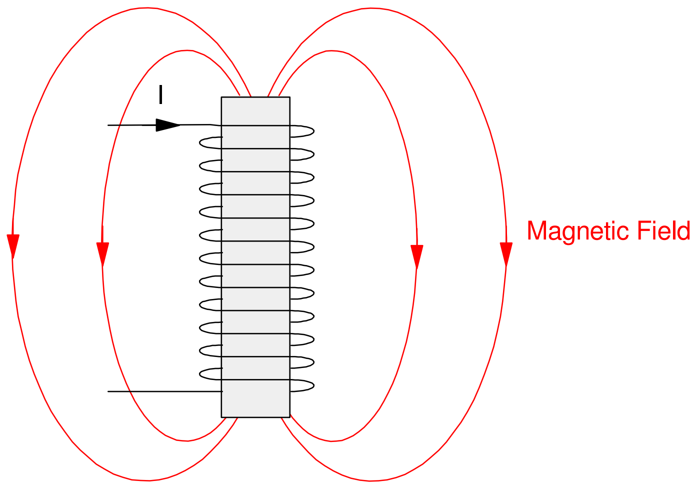

Figure 1: Inductors: Coils of wire produce a magnetic field, which in turns produces an inductor

In general, inductors are avoided if at all possible:

- Unlike electric fields and capacitors, magnetic fields do not have starting point and an ending point. That means that the magnetic fields extend outside the inductor and affect other components in a circuit.
- The core of an inductor almost has to be iron. Iron is about the only material which is strongly magnetic. This dependence upon iron makes inductors heavy, lossy, and only capable of supporting a limited amount of current until the iron saturates.
- The large amount of windings necessary to make an inductor creates a large resistive element with inductors, also reducing the efficiency.

Sometimes, you just can't avoid using inductors:

- Speakers rely upon electromagnets to work. An electromagnet is inherently an inductor.
- Motors rely upon electromagnetics to work. Again, any time you have an electromagnet you inherently have an inductor.
- Transmission lines are wires that carry electricity large distances. Current through a wire creates a magnetic field. This magnetic field stores energy, and again, created inductance.

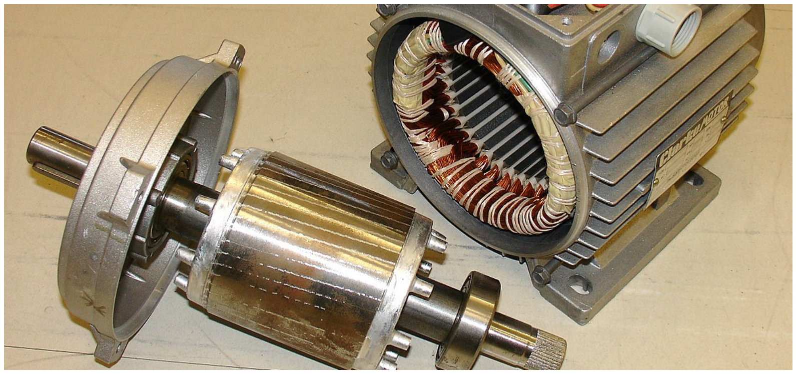

Figure 2: Motors rely on magnetic fields to work. Inductance is likewise unavoidable.

Example 1: Determine the inductance and resistance of an inductor like the one shown in Figure 1. Assume

- 100 windings of 36 gage wire copper wire (1.38 Ohms / meter)[1]
- Cross sectional area = 25mm^2
- Iron core with relative permeability of 800

Solution: From Electronics Tutorials[2]

$$L = \mu N^2 A / l$$

$$L = \left(4\pi \cdot 10^{-7} \tfrac{H}{m}\right)(800)(100)^2 (25 \cdot 10^{-6} m^2)/(0.025m)$$

$$L = 0.0101H$$

The resistance is

$$R = (100 \text{ windings})\left(0.02 \tfrac{\text{m}}{winding}\right)(1.38 \tfrac{\Omega}{m})$$

$$R = 2.77\Omega$$

This shows up in inductors you can get from Digikey:

- If you want large inductance, you have high resistance
- If you want low resistance, you get low inductance
- If you want a small inductor, you get both: low inductance and high resistance

1 https://www.engineeringtoolbox.com/copper-wire-d_1429.html

2 https://www.electronics-tutorials.ws/inductor/inductor.html

CX24	5250-RC
	
1 H	0.0001H
50 Ohms	0.216 Ohms
240mA (max)	2 A
$10.23 ea	$2.11

Table 1: Example of two inductors from Digikey. Resistance is unavoidable when using an inductor

Any time you produce a magnetic field, you get inductance. Electricity traveling through a wire, such as transmission lines, produce magnetic fields. Likewise, inductance is inherent with transmission lines as well.

For example, assume a high-voltage transmission line, Find the inductance.

- Length = 1km
- radius = 1cm
- frequency = 60Hz

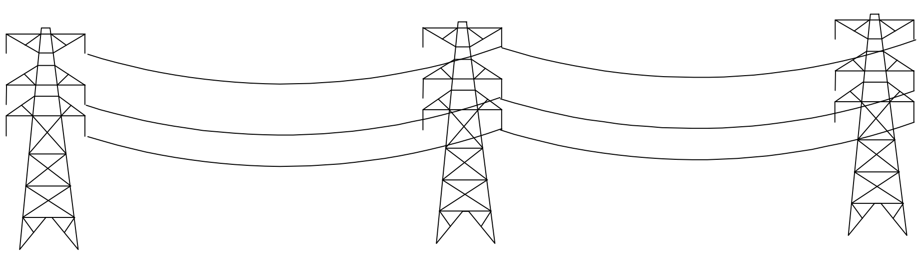

Figure 3: Long distance transmission lines produce magnetic fields - and hence have inductance.

From Wikipedia[3]

$$L = \frac{\mu_0}{2\pi} l (A - B + C)$$

$$A = \ln\left(\frac{l}{r} + \sqrt{\left(\frac{l}{r}\right)^2 + 1}\right) = 12.20$$

3 https://en.wikipedia.org/wiki/Inductor#Inductance_formulas

$$B = \frac{1}{\frac{r}{l}+\sqrt{1+\left(\frac{r}{l}\right)^2}} \approx 1$$

$$C = \frac{1}{4+r\sqrt{\frac{2}{\rho}\omega\mu}} = 0.1569$$

giving

$$L = 2.30\ \frac{mH}{km}$$

The inductance isn't a lot, but when a transmission line travels thousands of kilometers, it adds up.

VI Characteristics for an Inductor

The energy stored in the magnetic field for an inductor is

$$E = \frac{1}{2}LI^2$$

Taking the derivative gives you power

$$P = VI = \frac{d}{dt}\left(\frac{1}{2}LI^2\right)$$

$$VI = LI \cdot \frac{dI}{dt}$$

This gives the VI characteristic for an inductor

$$\boxed{V = L\frac{dI}{dt}}$$

What this tells you is

- Inductors are differentiators: the voltage across an inductor is proportional to the derivative of the current.
- Each inductor in a circuit adds a 1st-order differential equation.

You likewise need differential equations to describe circuits which include inductors.

In the following examples, we'll look at

- Finding the current vs. time for a circuit with a single inductor (1st-order differential equation), and
- Finding the current vs. time for a circuit with two inductors (2nd-order differential equation).

Single Inductor Circuit

Let's start with a circuit with a single inductor as in Figure 4. V0 is a step input

- V0(t) = 0V for t < 0
- V0(t) = 10V for t>0

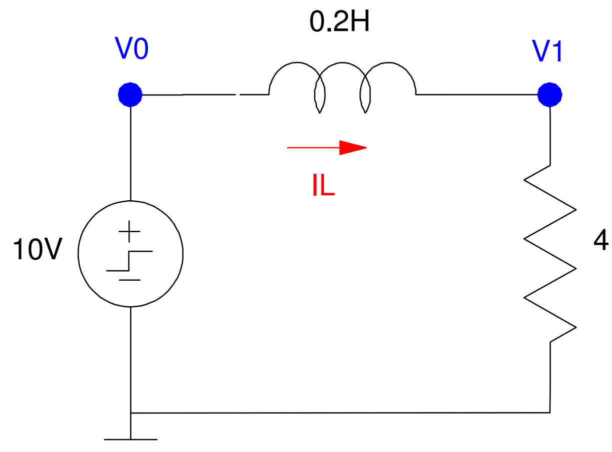

Figure 4: Find the current when V0 turns on at t=0

Since this circuit has a single inductor, a single differential equation is required to describe it. The equations for this circuit are

$$V_0(t) = 10\,u(t)$$

$$V_1 = 4I_L$$

$$V_1 = 0.2\frac{dI_1}{dt} = V_0 - V_1$$

Substituting

$$0.2\frac{dI_L}{dt} = V_0 - 4I_L$$

or

$$\frac{dI_L}{dt} = 5V_0 - 20I_L$$

Once you know IL(t), you can find V1(t), In Matlab. we can find V1 and IL using numerical integration

```
V0 = 10;
IL = 0
dt = 0.0025;
t = 0;
y = [];

while(t < 0.5)
   VL = V0 - 4*IL;
   y = [y ; [t, IL, VL] ];
   dIL  = 5*V0 - 20*IL;
   IL = IL + dIL * dt;
   t = t + dt;

   end

plot(y(:,1), y(:,2:3))
```

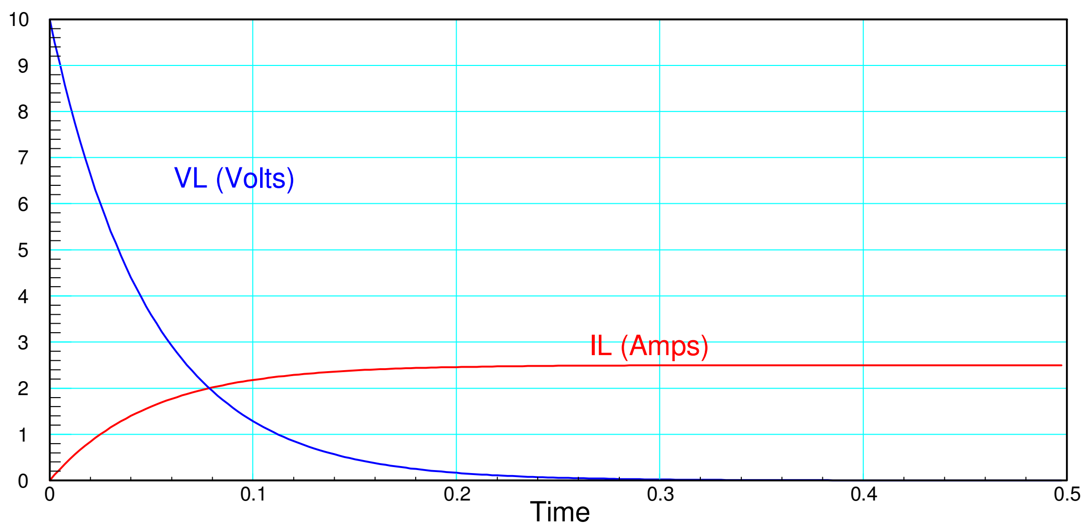

Figure 5: Voltage and current for a single-stage RL circuit

This result can be confirmed using CircuitLab. First, input the circuit into CircuitLab as in Figure 6. Note that V0 needs to be a step input rather than a constant voltage source

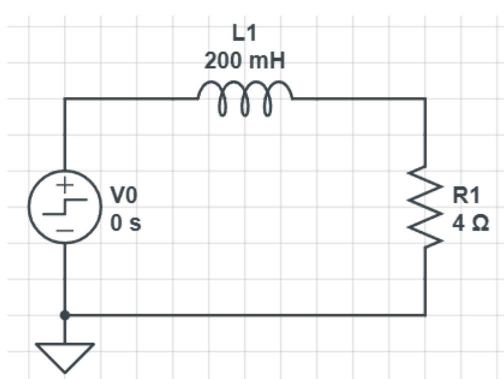

Figure 6: CircuitLab circuit to find the voltage and current for t>0

If you run a time-domain simulation, you get the same results as Matlab.

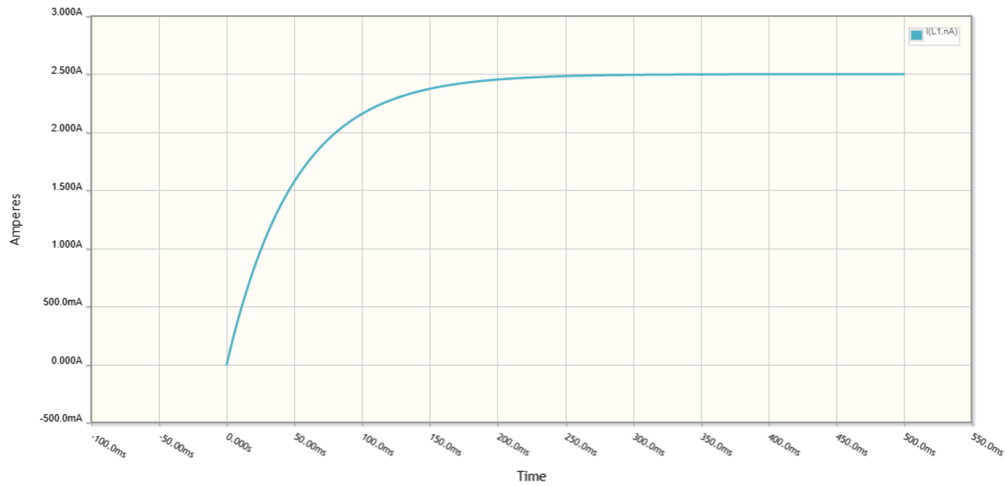

Figure 7: Voltage V1(t) for t>0 from a CircuitLab time-domain simulation

Note that CircuitLab does the same thing we did with Matlab:

- It finds the differential equation that describes the circuit.
- It then uses numerical integration to find the voltages and currents over time.

Two-Inductor Circuit

Next, consider a circuit with two inductors as in Figure 8. Note that each inductor adds a 1st-order differential equation. Likewise, with two inductors, a 2nd-order differential equation is required to describe this circuit.

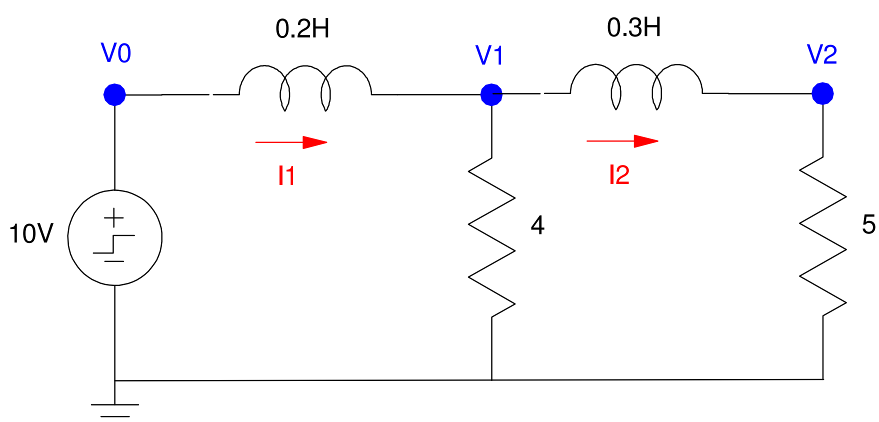

Figure 8: Circuit with two inductors. This requires a second-order differential equation to describe it.

There are several ways to write the differential equations for this circuit. One way is to first define the currents through the inductors as shown in Figure 8. From

$$V = L\frac{dI}{dt}$$

you can then write two coupled differential equations, one for each inductor. Start with

$$V_1 = 4(I_1 - I_2)$$

$$V_2 = 5I_2$$

Use the inductor equations

$$0.2\frac{dI_1}{dt} = V_0 - V_1$$

$$0.3\frac{dI_2}{dt} = V_1 - V_2$$

Substituting

$$0.2\frac{dI_1}{dt} = V_0 - 4(I_1 - I_2)$$

$$0.3\frac{dI_2}{dt} = 4(I_1 - I_2) - (5I_2)$$

Simplify

$$\frac{dI_1}{dt} = 5V_0 - 20I_1 + 20I_2$$

$$\frac{dI_2}{dt} = 13.333I_1 - 30I_2$$

Note that since inductors are current devices, it's actually easier if you write the differential equations in terms of current rather than voltage.

Once you have the differential equations, they can be solved using Matlab.

```
V0 = 10;
I1 = 0;
I2 = 0;
dt = 0.005;
t = 0;
y = [];

while(t < 1)
   y = [y ; [I1, I2] ];
   dI1 = 5*V0 - 20*I1 + 20*I2;
   dI2 = 13.333*I1 - 30*I2;
   I1 = I1 + dI1*dt;
   I2 = I2 + dI2*dt;
   t = t + dt;
   end

t = [1:length(y)]' * dt;
plot(t, y)
```

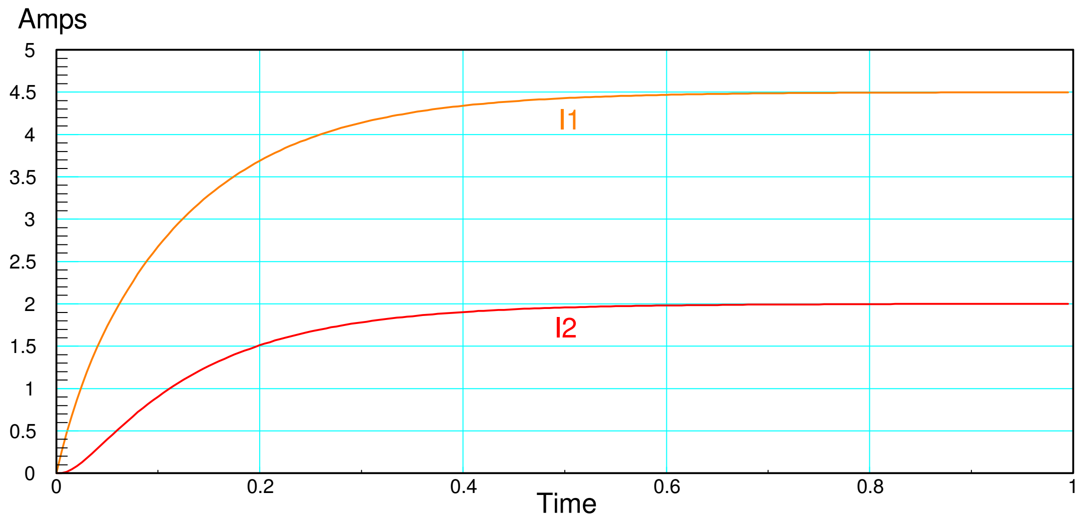

Figure 9: Current for t>0 found using numerical integration Matlab

This can be confirmed using CircuitLab. Start with building the circuit as shown in Figure 10. Note that V0 is a step function

- V0(t) = 0 for t < 0
- V0(t) = 10V for t > 0

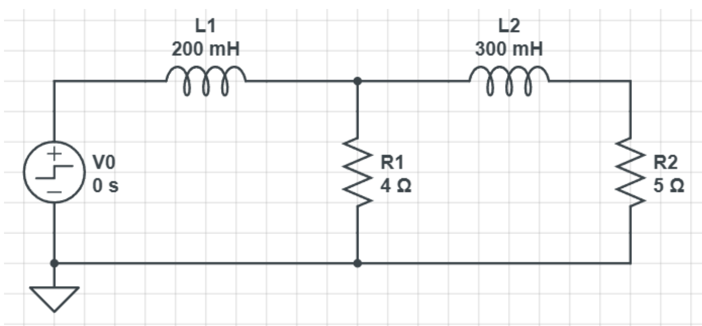

Figure 10 CircuitLab circuit for the two-inductor circuit

The currents for t>0 can then be found a time-domain simulation in CircuitLab. Note that the results in Figure 11 match up with the calculations in Matlab - confirming the Matlab solution.

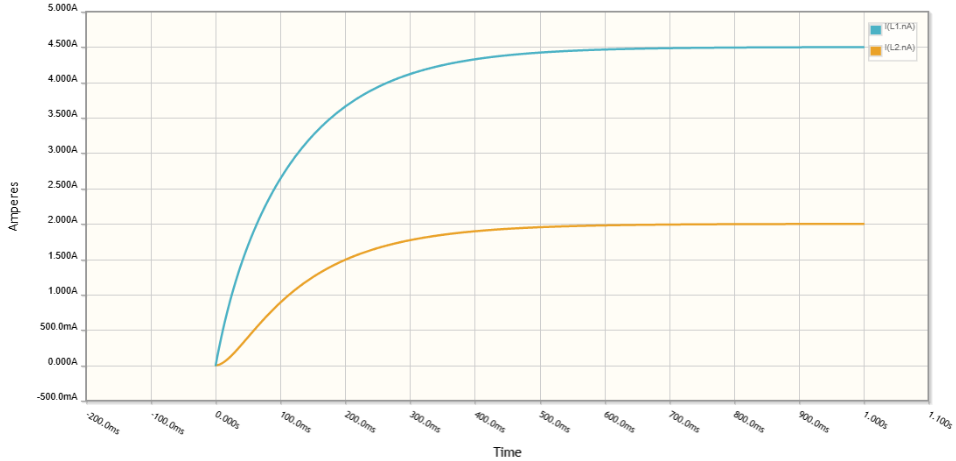

Figure 11: Currents for the two-inductor circuit found using CircuitLab and a time-domain simulation.

Eigenvalues and Eigenvectors

The previous simulation assumed zero initial conditions along with a forcing function, V0(t). This is terms the *forced response*. If instead you turn off the forcing function and instead provide initial conditions on I1 and I2, you get the *natural response*.

In Matlab, this can be done with a simple code change. If you want to find the natural response when

$$I_1(0) = I_2(0) = 1A$$

the Matlab code would be

```
V0 = 0;
I1 = 1;
I2 = 1;
dt = 0.005;
t = 0;
y = [];

while(t < 1)
   y = [y ; [I1, I2] ];
   dI1 = 5*V0 - 20*I1 + 20*I2;
   dI2 = 13.333*I1 - 30*I2;
   I1 = I1 + dI1*dt;
   I2 = I2 + dI2*dt;
   t = t + dt;
   end

t = [1:length(y)]' * dt;
plot(t, y)
```

The resulting currents vs. time are presented in Figure 12.

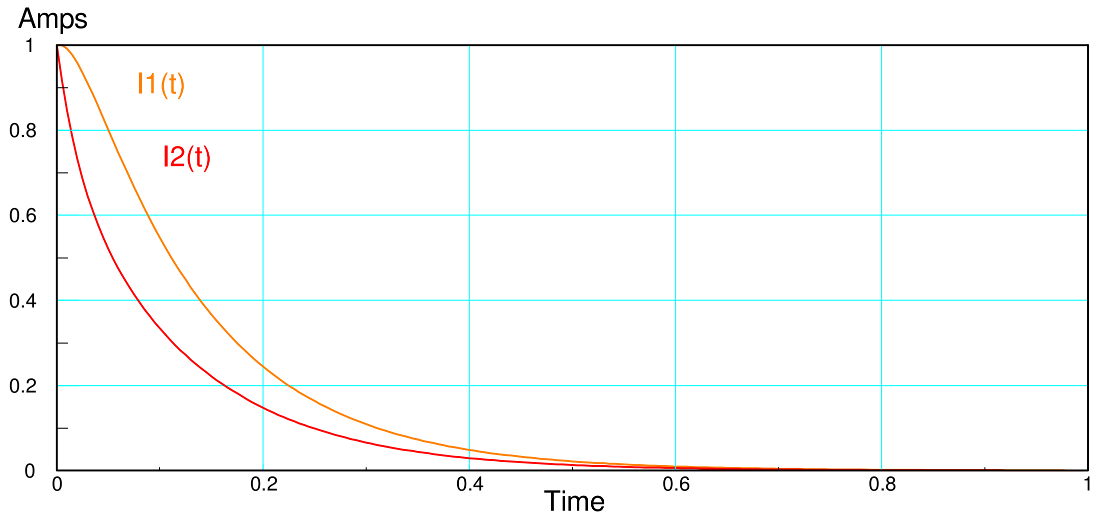

Figure 12: Natural response when I1(0) = I2(0) = 1A

This brings up a question. Assume I1(0) = 1A. What initial condition for I2(0) makes I2(t)

- Decay as fast as possible?
- Decay as slow as possible?
- How fast is that?

This is an eigenvalue / eigenvector problem.

Going back to the differential equations that describe this circuit

$$\frac{dI_1}{dt} = 5V_0 - 20I_1 + 20I_2$$

$$\frac{dI_2}{dt} = 13.333I_1 - 30I_2$$

This can be rewritten in matrix form

$$\begin{bmatrix} I_1' \\ I_2' \end{bmatrix} = \begin{bmatrix} -20 & 20 \\ 13.333 & -30 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \end{bmatrix} V_0$$

$$I' = AI + BV_0$$

Matrix A is a 2x2 matrix. It likewise has two eigenvalues and eigenvectors. These can be found in Matlab

```
>> A = [-20, 20 ; 13.33333, -30]

  -20.0000   20.0000
   13.3333  -30.0000
```

```
>> [eigenvector, eigenvalue] = eig(A)

eigenvector =

    slow     fast
    0.8560   -0.6714
    0.5170    0.7411


eigenvalue =

    slow     fast
  -7.9218          0
        0   -42.0782
```

The eigen values tell you *how* the system behaves

- The slow mode decays as exp(-7.9218t)
- The fast mode decays as exp(-42.0782t)

The eigenvector tells you *what* behaves that way. In general the response is composed of both the fast and slow mode:

$$I(t) = a \cdot \begin{bmatrix} 0.8560 \\ 0.5170 \end{bmatrix} \cdot e^{-7.9218t} + b \cdot \begin{bmatrix} -0.6714 \\ 0.7411 \end{bmatrix} \cdot e^{-42.0782t}$$

If the initial condition is proportional to the fast eigenvector, I(t) will decay slowly

$$I(0) = b \cdot \begin{bmatrix} -0.6714 \\ 0.74111 \end{bmatrix}$$

In Matlab

```
V0 = 0;
b = 1 / (-0.6714);
I1 = -0.6714 * b;
I2 = 0.7411  * b;
dt = 0.005;
t = 0;
y = [];

while(t < 1)
   y = [y ; I1 ];
   dI1 = 5*V0 - 20*I1 + 20*I2;
   dI2 = 13.333*I1 - 30*I2;
   I1 = I1 + dI1*dt;
   I2 = I2 + dI2*dt;
   t = t + dt;
   end

t = [1:length(y)]' * dt;
plot(t, y)
```

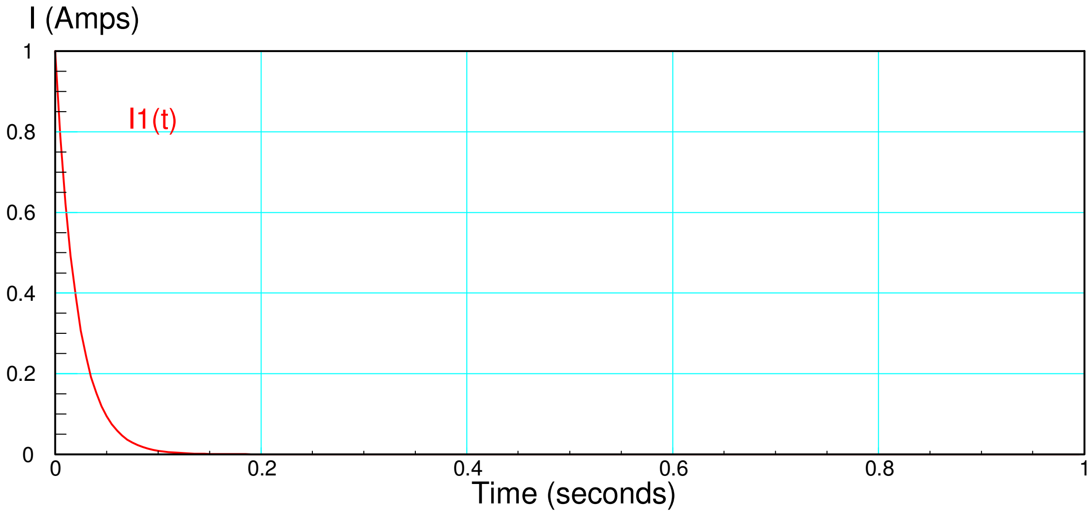

Figure 13: I1(t) when I2(t) excites the fast eigenvector

In contrast, if the initial condition is proportional to the slow eigenvector,

$$I(0) = a \cdot \begin{bmatrix} 0.8560 \\ 0.5170 \end{bmatrix}$$

then only the slow mode is excited. In Matlab

```
V0 = 0;
a = 1 / 0.8560;
I1 = 0.8560 * a;
I2 = 0.5170  * a;
dt = 0.005;
t = 0;
y = [];

while(t < 1)
   y = [y ; I1 ];
   dI1 = 5*V0 - 20*I1 + 20*I2;
   dI2 = 13.333*I1 - 30*I2;
   I1 = I1 + dI1*dt;
   I2 = I2 + dI2*dt;
   t = t + dt;
   end

t = [1:length(y)]' * dt;
plot(t, y)
```

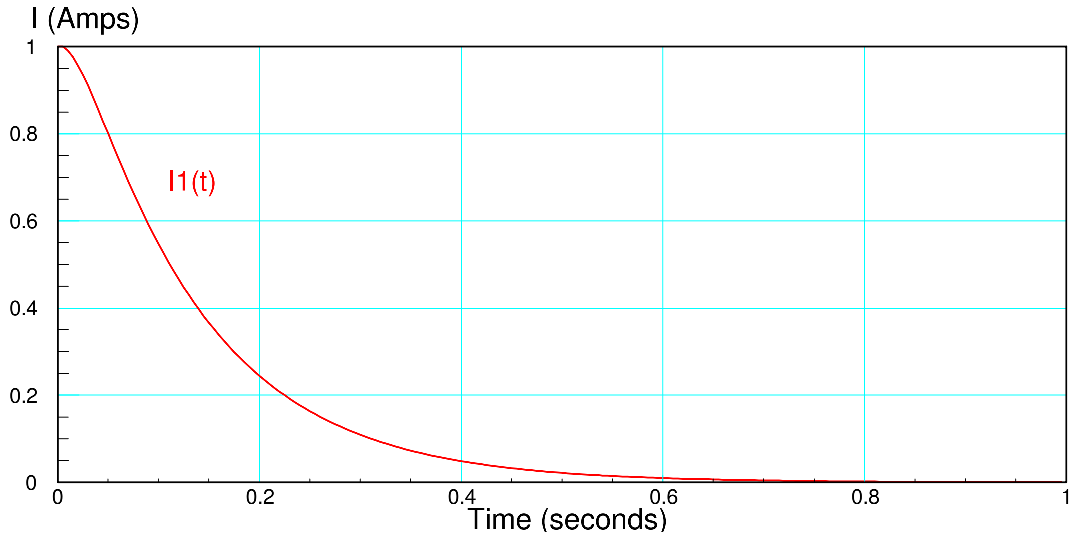

Figure 14: I1(t) when I2(0) excites only the slow eigenvector

If you have some other initial condition, such as

- I1(0) = 0A
- I2(0) = 1A

both modes are excited, The initial conditions determine a and b. In general

$$I(t) = a\begin{bmatrix} -0.6714 \\ 0.7414 \end{bmatrix} \cdot \exp(-42.08t) + b\begin{bmatrix} 0.8560 \\ 0.5170 \end{bmatrix} \cdot \exp(-7.92t)$$

Plugging in the initial conditions give

$$I(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = a\begin{bmatrix} -0.6714 \\ 0.7414 \end{bmatrix} + b\begin{bmatrix} 0.8560 \\ 0.5170 \end{bmatrix}$$

Solving gives

a = -0.06

b = -0.06

This shows up in Matlab with both the fast and slow mode being excited.

```
V0 = 0;
I1 = 0;
I2 = 1;
dt = 0.005;
t = 0;
y = [];

while(t < 1)
   y = [y ; I1 ];
   dI1 = 5*V0 - 20*I1 + 20*I2;
   dI2 = 13.333*I1 - 30*I2;
   I1 = I1 + dI1*dt;
   I2 = I2 + dI2*dt;
   t = t + dt;
   end

t = [1:length(y)]' * dt;
plot(t, y)
```

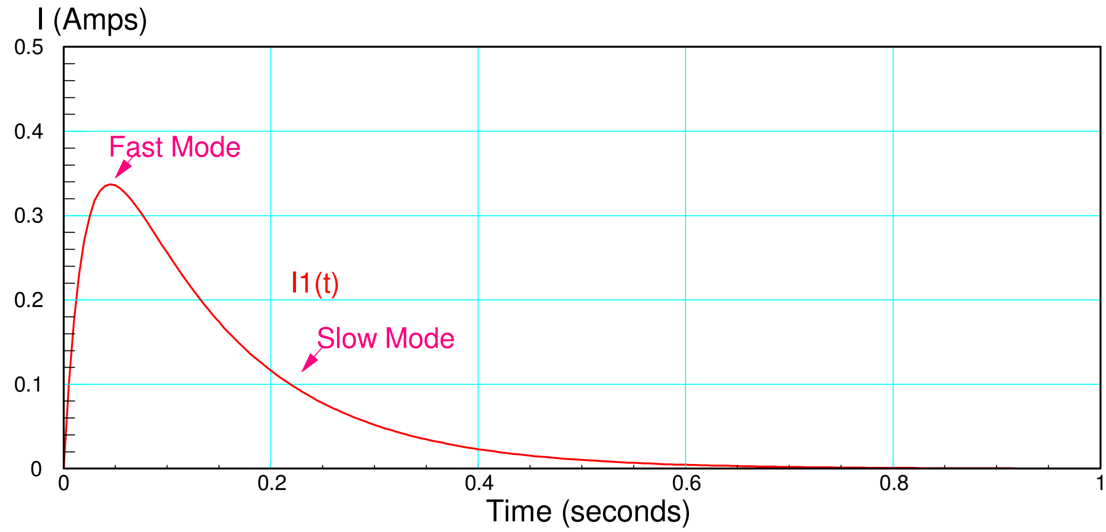

Figure 15: I1(t) when both the fast and slow modes are excited

Note in Figure 15 that

- Initially, both modes are excited.
- The fast mode decays quickly,
- Leaving the slow mode

Voltage Amplification

Switching gears a little. With inductors, the energy stored is a function of the current through an inductor. This means that current will always be a continuous function. Voltage, on the other hand, is a function of both current and resistance. If you can change the resistance connected to an inductor, you can actually amply voltage. You can even invert voltage - taking a +5V signal and making it -5V. This is often times the basis of a DC to DC converter.

For example, take the circuit of Figure 16. Assume that R is either 5 Ohms or 1000 Ohms (you can do this by connecting or disconnecting a 5 Ohm resistor in parallel with a 1000 Ohm resistor).

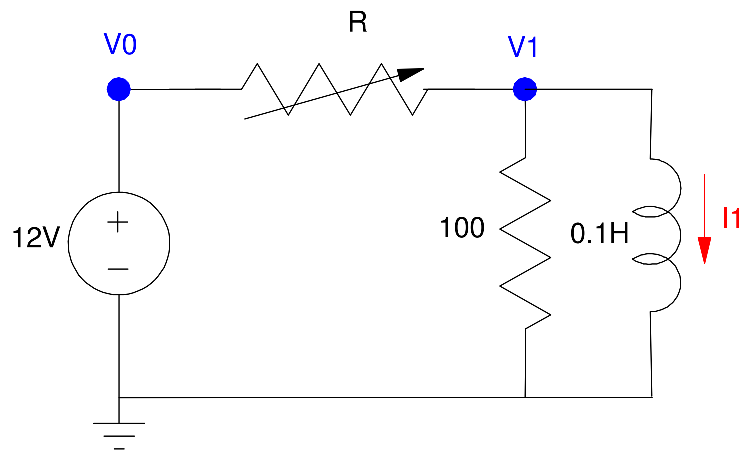

Figure 16: By varying R, the voltage at V1 can be increased and decreased.

The differential equation that describes this circuit is

Voltage node equation at V1

$$\left(\frac{V_1 - V_0}{R}\right) + \left(\frac{V_1}{100}\right) + I_1 = 0$$

along with the VI relationship for the inductor

$$V_1 = L\frac{dI_1}{dt}$$

Substituting

$$\left(\frac{1}{R} + \frac{1}{100}\right)L\frac{dI_1}{dt} = \left(\frac{V_0}{R}\right) - I_1$$

You can then solve for I1(t) using numerical integration. The voltage V1 is then related to I1(t) by its derivative.

In Matlab

```
V0 = 12;
I = 0
L = 0.1;
dt = 0.001;
t = 0;
Y = [];

while(t < 1)
   if(sin(6*pi*t) > 0)
       R = 5;
   else
       R = 20;
   end

   dI=(1/L)*(V0/R-I)/(1/R+1/100);

   I = I + dI * dt;
   t = t + dt;
   V1 = L*dI;

   Y = [Y ; [I, V1] ];
end

t = [1:length(Y)]' * dt;
plot(t,Y(:,2))
```

This results in the current through the inductor changing as R changes from 5 Ohms to 20 Ohms as presented in Figure 18. Note that current is a continuous function (energy cannot change in zero time).

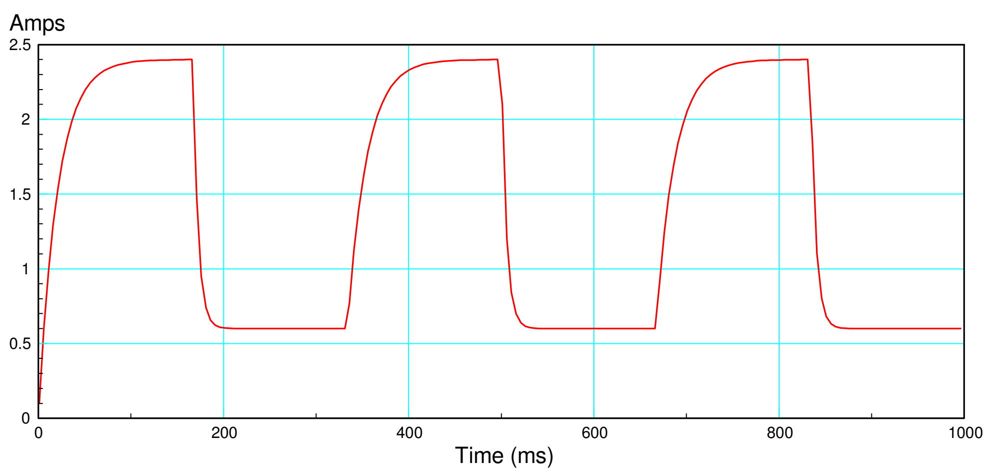

Figure 18: The inductor current varies as R changes from 5 Ohms to 20 Ohms

Voltage, on the other hand, can have jump discontinuities as R changes. In addition, voltage can be amplified and inverted if R changes suddenly as shown in Figure 19. This is useful for DC to DC converters where you need voltage amplification and/or inversion.

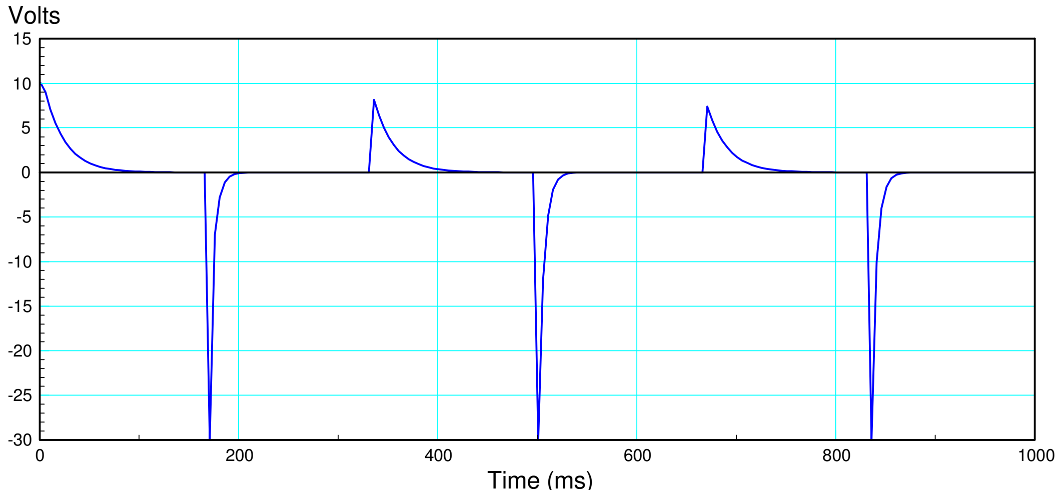

Figure 19: Voltage V1 varies from -30V to +10V as R varies. Note that this provides voltage inversion and amplification.

Summary

Inductors store energy in a magnetic field

$$E = \frac{1}{2}LI^2$$

The VI characteristics for an inductor are

$$V = L\frac{dI}{dt}$$

Each inductor in a circuit adds a 1st-order differential equation to the equations that describe a given circuit. The resulting currents for an RL circuit can be solved using

- CircuitLab, or
- Matlab with numerical integration, or
- Calculus (coming later)