Capacitors & The Heat Equation

Capacitors

A capacitor is a set of parallel plates[1] with the capacitance equal to

$$C = \varepsilon \frac{A}{d} \quad \text{(Farads)}$$

where

- ε is the dielectric constant of the material between plates (air = $8.84 \cdot 10^{-12}$)
- A is the area of the capacitor, and
- d is the distance between plates.

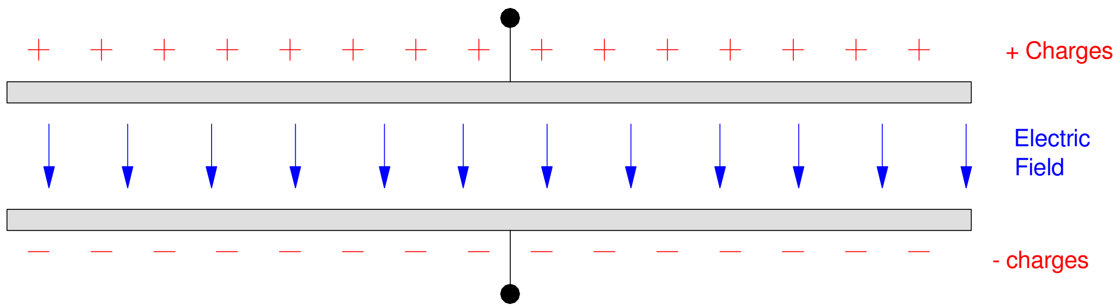

Figure 1: A capacitor is two parallel plates. They store energy in the electric field between the plates

The area you need for 1 Farad with plates 1mm apart is

$$1 = (8.84 \cdot 10^{-12}) \frac{A}{0.001m}$$

$$A = 113,122,171m^2$$

The capacitor would need to have dimensions of 10.6km x 10.6km for a capacitance of 1 Farad. Typically, capacitors are in the order if micro-farads.

The charge stored in a capacitor is proportional to the voltage as

$$Q = C \cdot V$$

where Q is the charge in Coulombs (one Coulomb is equal to $6.242 \cdot 10^{18}$ electrons). When the voltage across a capacitor drops, the charge stored drops proportionally. This gives the fundamental equation for a capacitor:

$$I = \frac{dQ}{dt} = C \frac{dV}{dt} + V \frac{dC}{dt}$$

Assuming the capacitance is constant

1 http://www.electronics-tutorials.ws/

$$\boxed{I = C\frac{dV}{dt}}$$

This means that capacitors are integrators:

$$V = \frac{1}{C}\int I \cdot dt$$

In Calculus, you will be covering integration and differentiation and how to come up with a closed-form solution to various problems. With MATLAB (i.e. in this class) you can solve using numerical methods.

Capacitors and Energy Storage

Capacitors store energy when charged. The energy stored is

$$P = VI = V \cdot C\frac{dV}{dt}$$

$$E = \int P\,dt = \int \left(VC\frac{dV}{dt}\right)dt = \int (VC)dV$$

$$E = \frac{1}{2}CV^2$$

The energy stored in a capacitor isn't large - but it is there. To put this in perspective, the energy stored in a 1F capacitor compared to other common items are as follows:

Item	Energy (Joules)	Cost	$ / MJ
1 pound Wyoming Coal	3,600,000	$0.028	$0.0078
1 pound ND Lignite	1,565,217	$0.017	$0.0108
1 pint of gasoline	15,000,000	$0.37	$0.0247
Lithium battery (D cell)	246,240	$6	$24.4
1F Capacitor (5V)	12.5	$1.53	$122,400

Table 1: Cost per mega-joule of energy storage for various items

In theory, you *could* build an electric car using capacitors to store the energy. This would have advantages:

- It would take seconds to charge rather than hours,
- The efficiency of capacitors is near 100% - slightly better than chemical batteries.
- The number of charge / recharge cycles is large (almost unlimited),

But,

- Electric cars would cost 5,000 times more for the same energy storage.

Instead, capacitors are used in electronic circuits to provide energy for a short period of time - often on the order of milliseconds or microseconds. When dealing with such short time frames, even 1mJ can be significant. For example, 1mJ may not seem like a lot, but 1mJ dissipated offer 1us is 1000W

$$P = \frac{\text{Joules}}{\text{second}} = \frac{1mJ}{1\mu s} = 1000W$$

Numerical Integration and Capacitors

Capacitors are inherently integrators:

$$I = C\,\frac{dV}{dt}$$

$$V = \frac{1}{C}\int I\,dt$$

Likewise, any circuit which has capacitors implements a differential equation. To solve for the voltages, just integrate the current to each capacitor.

In Calculus, you studies methods to solve differential equations. In Circuits II and Signals and Systems, you will study LaPlace transforms - an easier way to solve differential equations. Here, in Circuits I, we will focus on

- Obtaining the differential equations that describe RC circuits, and
- Solving for the voltages using numerical methods and Matlab.

The latter is also how programs like CircuitLab solve these problems.

Numerical integration covers how to find the integral of a function using a computer program. There are many different forms of numerical integration, and all are off slightly. Three common forms are

- Euler Integration
- Trapezoid Rule
- Runge Kutta Integration

Euler Integration: Suppose you want to find the integral of x(t)

$$y(t) = \int x(t)\,dt$$

The integral is the area under the curve. One way to approximate this area is to

- Sample x(t) every T seconds,
- Use rectangles to approximate the area each T seconds, and
- Sum up the area of each rectangle.

Euler integration is the simplest and least accurate of these three forms. It's simple though and not too bad if you keep the sampling time (dt) small.

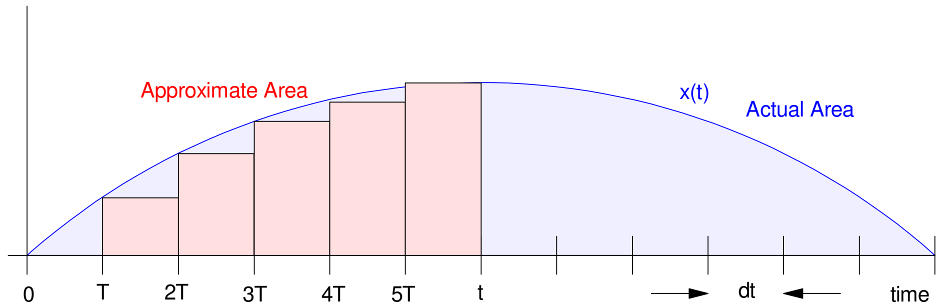

Figure 2: Euler Integration: Approximate the area under the curve with rectangles

The nice thing about Euler Integration is it's simple and requires no memory. The integral at time t is

$$y(t) = y(t-T) + x(t) \cdot dt$$

Trapezoid (Bilinear) Integration: A significantly improved version of integration approximates the area under the curve with trapezoids. This is slightly more complicated in that you need to remember the previous value of x(t):

$$y(t) = y(t-T) + \left(\frac{x(t)+x(t-T)}{2}\right) \cdot dt$$

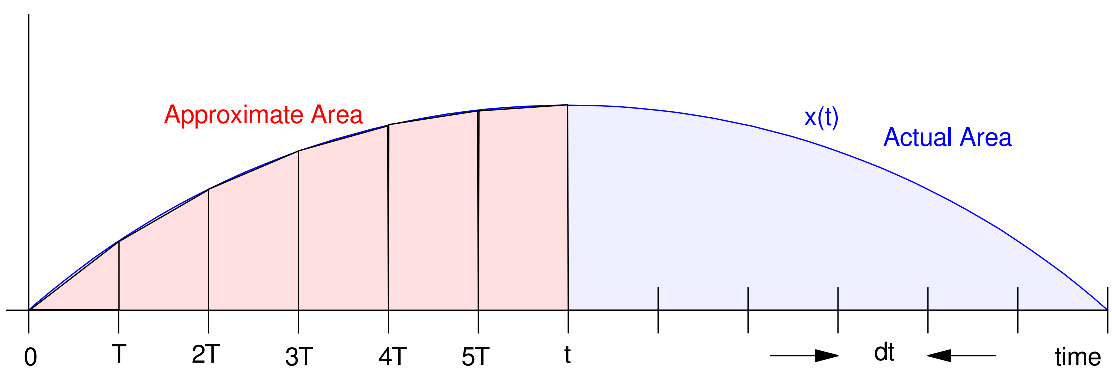

Figure 3: Trapezoid Integration: Approximate the area under the curve with trapezoids

Runge Kutta Integration: Even better results are obtained if you also include points within each sample and curve fit a parabola to the curve, and compute the area under the parabola (3rd order Runge Kutta) or a cubic (4th order Runge Kutta), or 4th-order polynomial (5th order Runge Kutta).

Here, we will stick with Euler Integration, meaning

To find the voltage across a capacitor

- Compute the current to the capacitor, and
- Integrate using Euler integration:

```
dV = I / C
V = V + dV * dt
```

Example 1: 1-Stage RC Circuit

Assume

$$V_0(t) = 10u(t) = \begin{cases} 0V & t < 0 \\ 10V & t > 0 \end{cases}$$

Find V1(t)

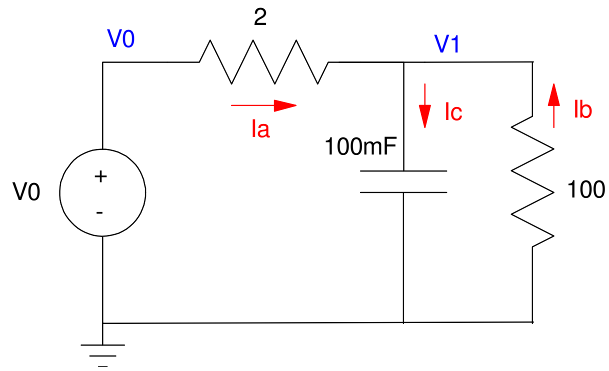

Figure 4: One-stage RC circuit

Solution: V1 is

$$V_1(t) = \frac{1}{C}\int I_c\, dt$$

$$I_c = C\,\frac{dV_1}{dt}$$

To find V1, you first have to find Ic

$$I_c = I_a + I_b$$

$$I_c = \left(\frac{V_0 - V_1}{2}\right) + \left(\frac{0 - V_1}{100}\right)$$

$$\frac{dV}{dt} = \frac{1}{C} I_c$$

meaning

$$\frac{dV_1}{dt} = -5.1\,V_1 + 5\,V_0$$

In Matlab, you can integrate and plot V1 vs time:

```
% 1-stage RC Filter

V = 0;
V0 = 10;
dt = 0.01;
t = 0;
Y = [];

while(t < 1)
   dV  = -5.1*V + 5*V0;
   V = V +  dV*dt;
   t = t + dt;
   Y = [Y ; V];
end

t = [1:length(Y)]' * dt;

plot(t, Y);
```

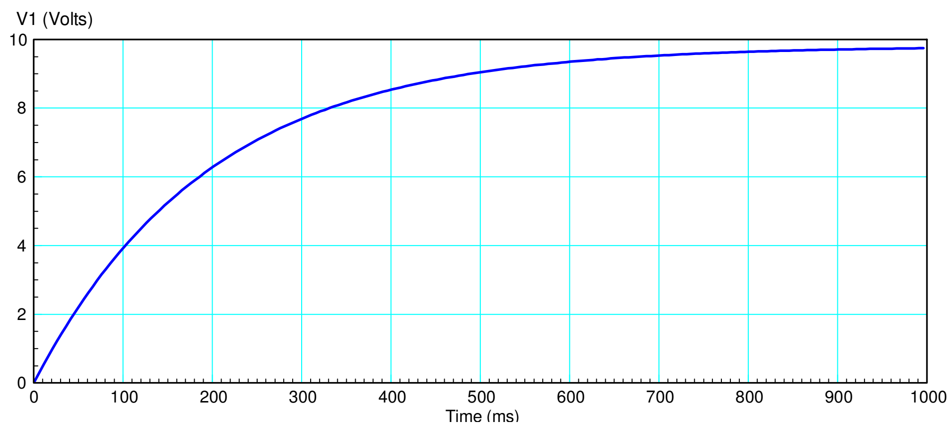

Figure 5: Voltage V1 vs. Time
Note that this shape is very typical of RC filters: you are charging up the capacitor

In CircuitLab, you can generate the same plot with CircutLab

- Change the input to a square wave, 0.1Hz, 0.1 degree phase shift
- Run a time-domain simulation for 1 second

CircuitLab will run the same numerical integration scheme we did in Matlab (except probably with trapezoid rule or Runge Kutta integration).

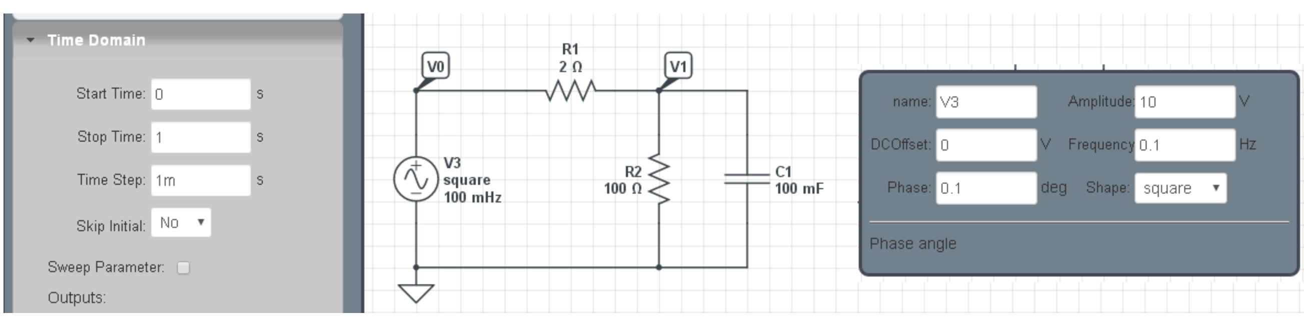

Figure 6: Simulating a 1-Stage RC Circuit in CircuitLab

Example 2: 2-Stage RC Filter

Consider next a two-stage RC filter as presented in Figure 7. Assume

- V0 = 0V t < 0
- V0 = 10V t > 0

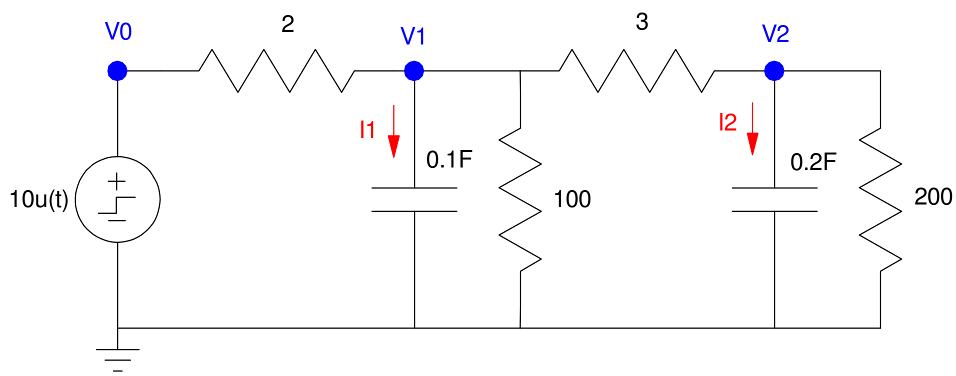

Figure 7: Two-stage RC filter. Find V2(t) when V0 turns on at t=0

To find V2(t), wtart by writing the differential equations that describe this circuit. Since it has two capacitors, two coupled first-order differential equations are needed. From

$$I = C\frac{dV}{dt}$$

you get

$$I_1 = 0.1\frac{dV_1}{dt} = \left(\frac{V_0 - V_1}{2}\right) + \left(\frac{V_2 - V_1}{3}\right) - \left(\frac{V_1}{100}\right)$$

$$I_2 = 0.2\frac{dV_2}{dt} = \left(\frac{V_1 - V_2}{3}\right) - \left(\frac{V_2}{200}\right)$$

Simplifying gives

$$\frac{dV_1}{dt} = 5V_0 - 8.433V_1 + 3.33V_2$$

$$\frac{dV_2}{dt} = 1.667V_1 - 1.692V_2$$

These can be solved in Matlab using numerical integration

```
V1 = 0;
V2 = 0;
V0 = 10;
t = 0;
dt = 0.01
y = [];
while(t < 5)
    dV1 = 5*V0 - 8.433*V1 + 3.33*V2;
    dV2 = 1.667*V1 - 1.692*V2;

    V1 = V1 +dV1*dt;
    V2 = V2 + dV2*dt;
    t = t + dt;
```

```
    y = [y ; t, V0, V1, V2];
end

plot(y(:,1),y(:,2:4))
```

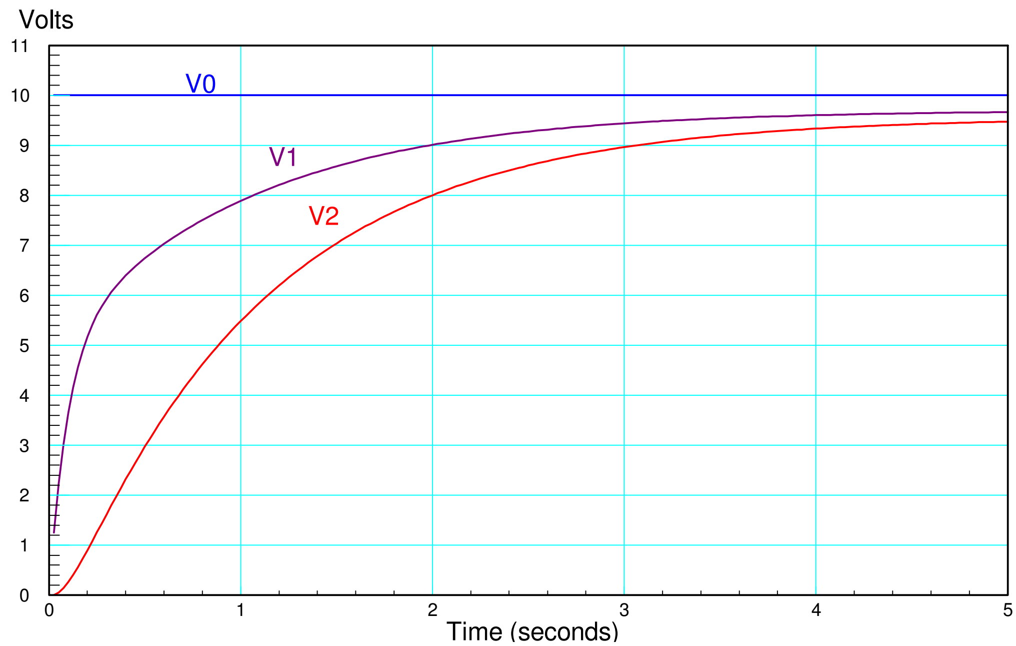

Figure 8: Voltages vs. time solved using numerical integration

If you run a time-domain simulation in CircuitLab, you get exactly the same results. When you run time-domain simulations, CircuitLab also uses numerical integration to solve the same differential equations we solved here.

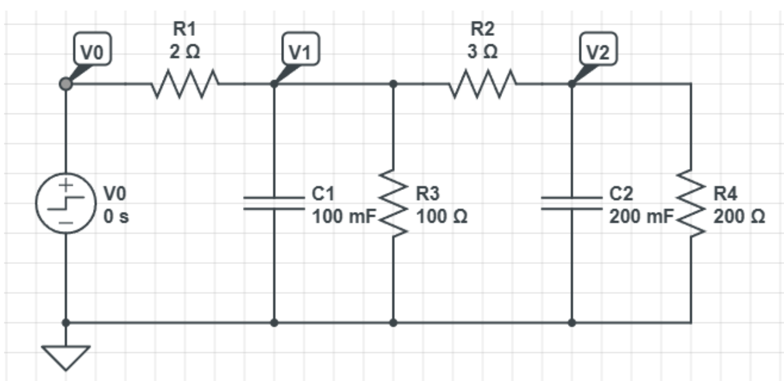

Figure 9: 2-stage RC filter input into CIrcuitLab

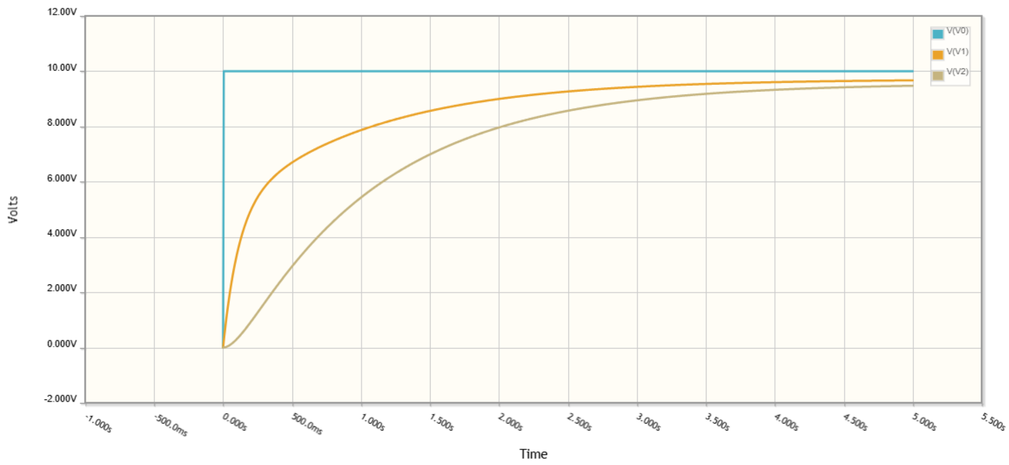

Figure 10: Result of a time-domain simulation in CIrcuitLab gives the same results as Matlab

Eigenvalues and Eigenvectors

Note that in the Matlab code, you could provide an input along with zero initial condistion as done before:

```
V1 = 0;
V2 = 0;
V0 = 10;
```

The results is termed the forced response. You could also find the voltages assuming some other initial condition along with zero input

```
V1 = 7;
V2 = 1;
V0 = 0;
```

This brings up a quesiton. Assume that V2(0) = 1. What value of V1

- Makes V2(t) decay as fast as possible?
- Makes V2(t) decay as slow as possible?
- How fast or slow is that?

This is an eigenvalue / eigenvector question.

Eigenvalues and eigenvectors are covered in Math 129: Linear Algebra. In that class, eigenvalues are defined as the solution to

$$|\lambda I - A| = 0$$

Eigenvectors are defined as

$$A\Lambda = \lambda\Lambda.$$

So what's that mean?

This circuit is a good way to help explain eigenvalues and eigenvectors.

Start with the differential equations that describe this circuit.

$$\frac{dV_1}{dt} = 5V_0 - 8.433V_1 + 3.33V_2$$

$$\frac{dV_2}{dt} = 1.667V_1 - 1.692V_2$$

In matrix form, this can be written as

$$\begin{bmatrix} \frac{dV_1}{dt} \\ \frac{dV_2}{dt} \end{bmatrix} = \begin{bmatrix} -8.4333 & 3.333 \\ 1.667 & -1.692 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \end{bmatrix} V_0$$

or

$$\frac{dV}{dt} = AV + BV_0$$

Matrix A is a 2x2 matrix. It likewise has two eigenvalues and two eigenvectors. These can be found using Matlab:

```
>> A = [-8.4333  3.3333 ; 1.667 , -1.692[

   -8.4333    3.3333
    1.6670   -1.6920

>> [Eigenvectors, Eigenvalues] = eig(A)

Eigenvectors =

   -0.9761   -0.4069
    0.2174   -0.9135

Eigenvalues =

   -9.1758         0
         0   -0.9495
```

The eigenvalues tell you *how* the system behaves:

- Something decays as exp(-9.1758t)
- Something else decays as exp(-0.9495t)

Or, put another way, the natural response of V2(t) has two terms:

$$V_2(t) = a_2 \cdot \exp(-0.9495t) + b_2 \cdot \exp(-9.1758t)$$

The eigenvectors tell you *what* decays that way. In general, any intial condition decays as

$$V(t) = a \cdot \Lambda_1 \cdot \exp(\lambda_1 t) + b \cdot \Lambda_2 \cdot \exp(\lambda_2 t)$$

For this system, voltages V1 and V2 decay as

$$V(t) = a \cdot \begin{bmatrix} -0.9761 \\ 0.2174 \end{bmatrix} \cdot \exp(-9.1758t) + b \cdot \begin{bmatrix} 0.4069 \\ 0.9135 \end{bmatrix} \cdot \exp(-0.9495t)$$

The initial condition tells you 'a' and 'b'.

$$V(0) = a \cdot \begin{bmatrix} -0.9761 \\ 0.2174 \end{bmatrix} + b \cdot \begin{bmatrix} 0.4069 \\ 0.9135 \end{bmatrix}$$

For example, if you make the intial condition propotioanl to the fast eigenvector, such as

$$V(0) = \left(\frac{1}{0.2147}\right)\begin{bmatrix} -0.9761 \\ 0.2147 \end{bmatrix} = \begin{bmatrix} -4.5463 \\ 1.000 \end{bmatrix}$$

then

$$a = \left(\frac{1}{0.2147}\right), \; b = 0$$

and

$$V(t) = \cdot \begin{bmatrix} -4.5463 \\ 1 \end{bmatrix} \cdot \exp(-9.1758t)$$

or

$$V_2(t) = \exp(-9.1758t)$$

ans: To make V2(t) decay quickly, make V1(0) -4.5463V

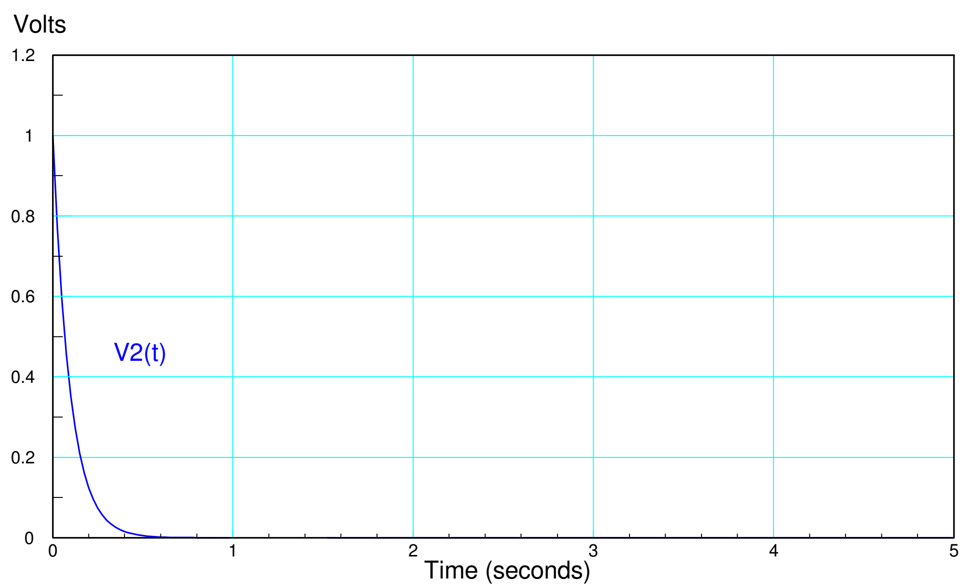

Figure 11: V2(t) when V1(t) excites the fast mode

On the other hand, if you make the intial condition proportional to the slow eigenvector, such as

$$V(0) = \left(\frac{1}{0.9135}\right)\begin{bmatrix} 0.4069 \\ 0.9135 \end{bmatrix} = \begin{bmatrix} 0.4454 \\ 1.000 \end{bmatrix}$$

then

$$a = 0,\ \ b = \left(\frac{1}{0.9135}\right)$$

and

$$V(t) = \begin{bmatrix} 0.4454 \\ 1.000 \end{bmatrix} \cdot \exp(-0.9495t)$$

or

$$V_2(t) = \exp(-0.9495t)$$

ans: To make V2(t) decay quickly, make V1(0) +0.4454V

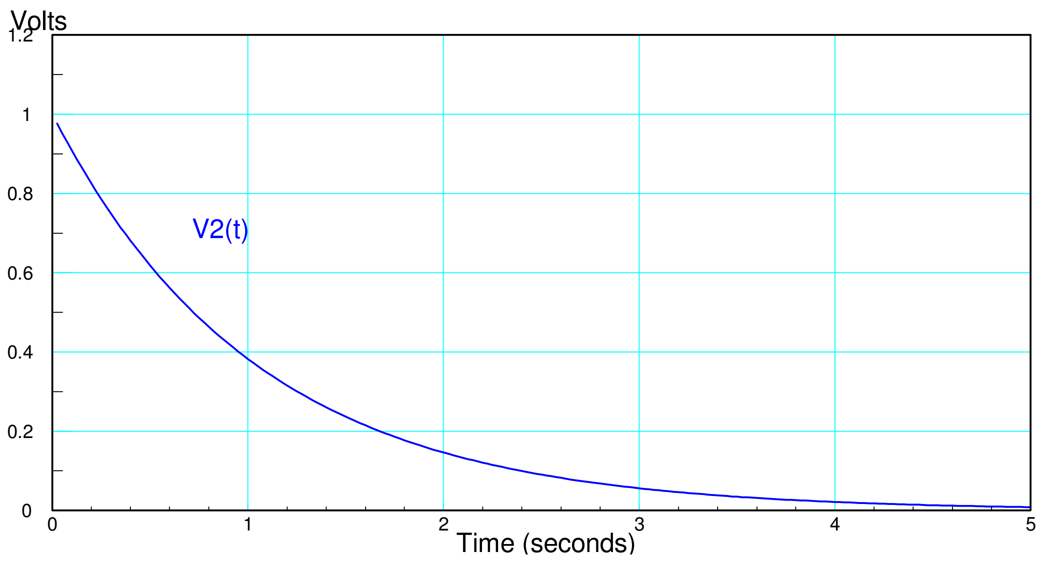

Figure 12: V2(t) when V1(t) excites only the slow mode

If you start with some other intial condition, such as

```
V1 = 1;
V2 = 1;
V0 = 0;
```

both modes are excited. The values of 'a' and 'b' can be found by inverting the eigecvector matrix

$$V(t) = a \cdot \begin{bmatrix} -0.9761 \\ 0.2174 \end{bmatrix} \cdot \exp(-9.1758t) + b \cdot \begin{bmatrix} 0.4069 \\ 0.9135 \end{bmatrix} \cdot \exp(-0.9495t)$$

$$V(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = a \cdot \begin{bmatrix} -0.9761 \\ 0.2174 \end{bmatrix} + b \cdot \begin{bmatrix} 0.4069 \\ 0.9135 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -0.5169 \\ 1.2177 \end{bmatrix}$$

This results in V2(t) being

$$V_2(t) = a \cdot 0.2174 \cdot \exp(-9.1758t) + b \cdot 0.9135 \cdot \exp(-0.9495t)$$

In Matlab

```
V1 = 1;
V2 = 1;
V0 = 0;
t = 0;
dt = 0.025
y = [];
while(t < 5)
    dV1 = 5*V0 - 8.433*V1 + 3.33*V2;
    dV2 = 1.667*V1 - 1.692*V2;

    V1 = V1 +dV1*dt;
    V2 = V2 + dV2*dt;
    t = t + dt;
    y = [y ; t, V0, V1, V2];
end

plot(y(:,1),y(:,2:4))
```

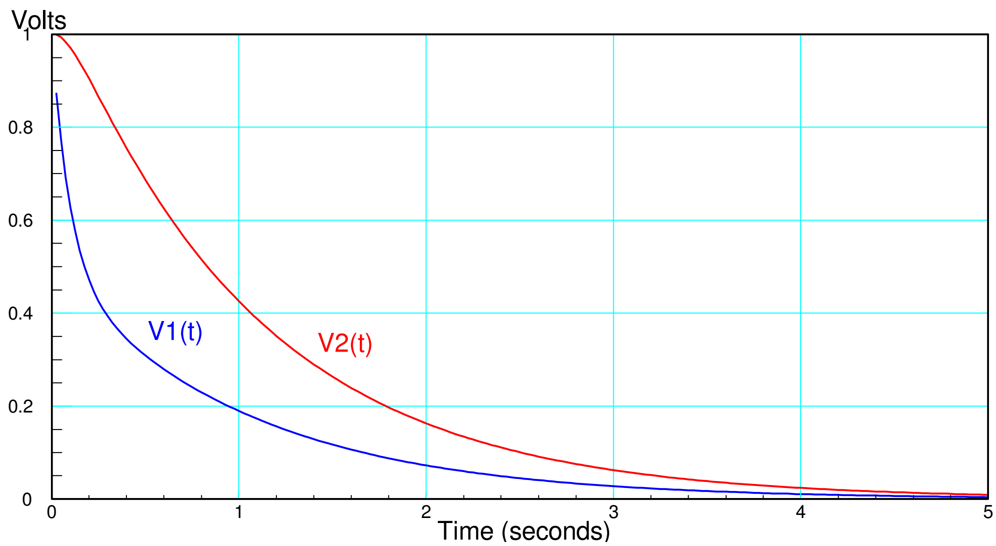

Figure 13: V1(2) and V2(t) when the initial conditions excite both modes

Summary

Capacitors are integrators

$$I = C\frac{dV}{dt}$$

$$V = \frac{1}{C}\int I(t) \cdot dt$$

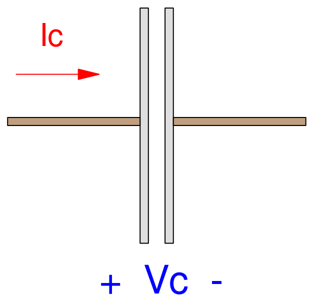

Figure 14: The voltage across a capacitor is the integral of the current flowing into the capacitor

Each capacitor adds a 1st-order differential equation to a circuit. To find the voltages with an RC circuit

- Use CircuitLab's time response, or
- Use numerical integration

Eigenvalues & eigenvectors tell you

- *How* the system behaves (eigenvalues)
- *What* behaves that way (eigenvectors)