Superposition

Introduction:

In pretty much all of the circuits covered so far, there has been only one input or one independent source. For some circuits, there could be several inputs or several independent sources. For example, a mixer circuit generates an output which is a combination of several inputs. Figure 1 presents another circuit with two inputs. As another example, a given input could contain several terms such as

$$V_0 = 2 + 3\sin(4t) + 5\cos(6t)$$

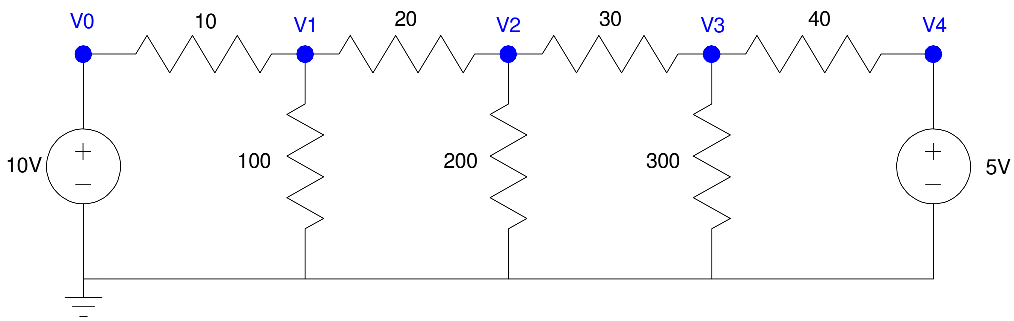

Figure 1: Circuit with two inputs

In this lecture, the notion of *superposition* is introduced. With superposition, circuits with multiple inputs can be treated as separate circuits, each with a single input. This can simplifies circuit analysis.

Superposition

A circuit composed of resistors, inductors, capacitors, voltage sources, current sources, and dependent sources is a linear system. Linear systems have the property

$$f(a+b) = f(a) + f(b)$$

This, in a nut-shell, is superposition. If you have multiple inputs to a circuit, you can analyze the circuit with one element at a time. The answer will be the sum of the two.

Example 1:

Consider the problem of finding the voltages of the circuit in Figure 1. One way to find the voltages is to write the voltage node equations then solving. For this circuit, the voltage node equations are:

$$V_0 = 10$$

$$\left(\frac{V_1 - V_0}{10}\right) + \left(\frac{V_1}{100}\right) + \left(\frac{V_1 - V_2}{20}\right) = 0$$

$$\left(\frac{V_2 - V_1}{20}\right) + \left(\frac{V_2}{200}\right) + \left(\frac{V_2 - V_3}{30}\right) = 0$$

$$\left(\frac{V_3 - V_2}{30}\right) + \left(\frac{V_3}{300}\right) + \left(\frac{V_3 - V_4}{40}\right) = 0$$

$$\left(\frac{V_4 - V_3}{40}\right) + \left(\frac{V_4}{400}\right) = 0$$

$$V_4 = 5$$

Solving 6 equations for 6 unknowns results in

Case	V0	V1	V2	V3	V4
V0 on, V4 on	10.00 V	8.42 V	6.95 V	5.78 V	5.00 V

Table 1: Voltages when V0 and V4 are both turned on

A second way to find these voltages is to use superposition:

- Case A: V0 is turned on and V4 is turned off
- Case B: V4 is turned on and V0 is turned off

By superposition, the total voltage at each node will he the sum of these results.

First, turn off V4 as shown in Figure 2:

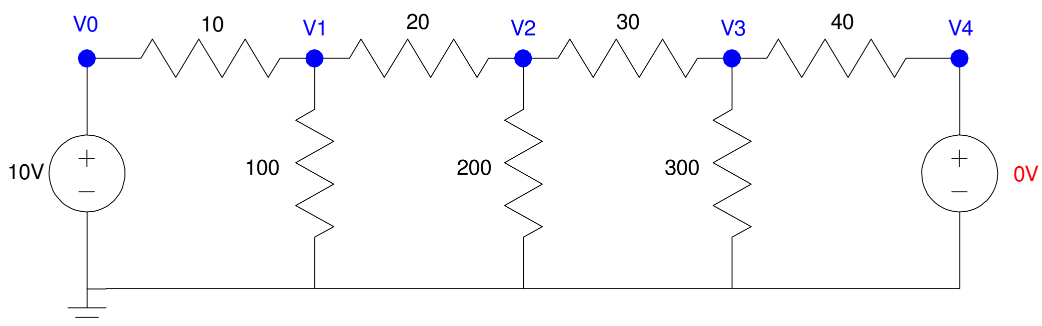

Figure 2: Case A: V0 is turned on and V4 is turned off.

Solving for the voltages results in

Case	V0	V1	V2	V3	V4
V0 on, V4 off	10.00V	8.04 V	5.71 V	3.09 V	0.00 V

Table 2: Voltages when V0 is on and V4 is off

Next, turn on V4 and turn off V0 as in Figure 3

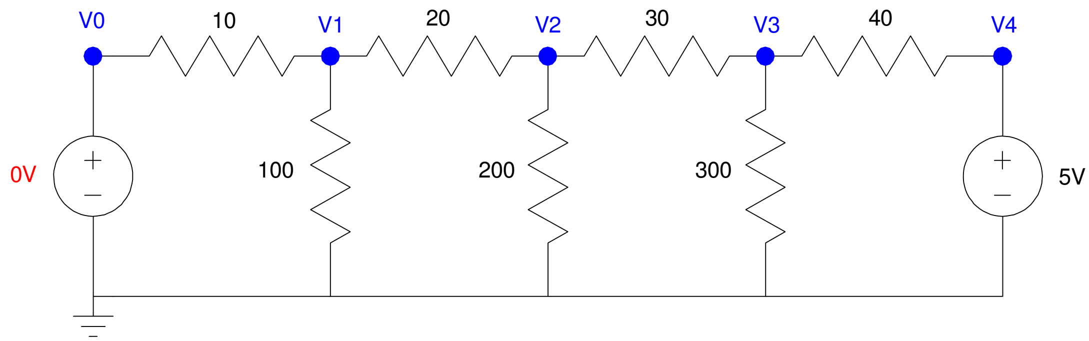

Figure 3: V0 is turned off and V4 is turned back on

The resulting voltages are as presented in Table 4

Case	V0	V1	V2	V3	V4
V0 on, V4 off	0.00 V	0.386 V	1.24 V	2.69 V	5.00 V

Table 3: Voltages when V0 is off and V4 is on

Finally, if you add these voltages together, you get the same results of Table 1, when both inputs were turned on.

Case	V0	V1	V2	V3	V4
V0 on, V4 off	10.00V	8.04 V	5.71 V	3.09 V	0.00 V
V0 off, V4 on	0.00 V	0.386 V	1.24 V	2.69 V	5.00 V
V0 on, V4 on	10.00 V	8.42 V	6.95 V	5.78 V	5.00 V

Table 4: Adding the voltages when V0 is on with when V4 is on gives the voltages when both are on.

Example 2: R-2R Ladder

Superposition can make circuit analysis easier. For example, consider an R-2R ladder as presented in Figure 4. From superposition, you know that the output is of the form

$$Y = aA + bB + cC + dD$$

where {a, b, c, d} are constants.

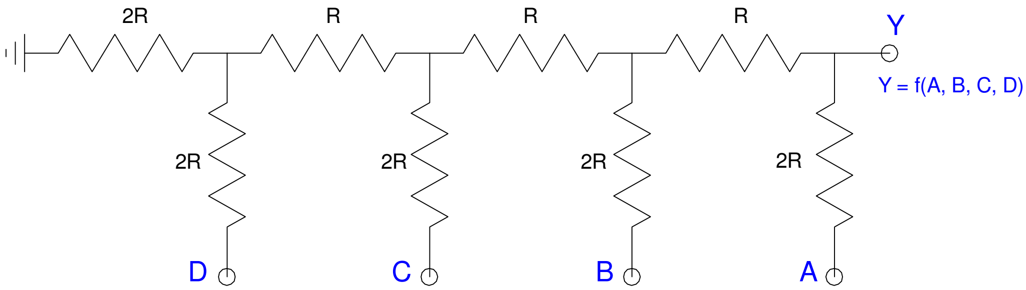

Figure 4: R - 2R Ladder

The value of each weighting can be found using superposition.

Weighting 'a': To find the weighting for input A,

- Set the voltage at A equal to 1V
- Set the other inputs to 0V

This is shown in Figure 5.

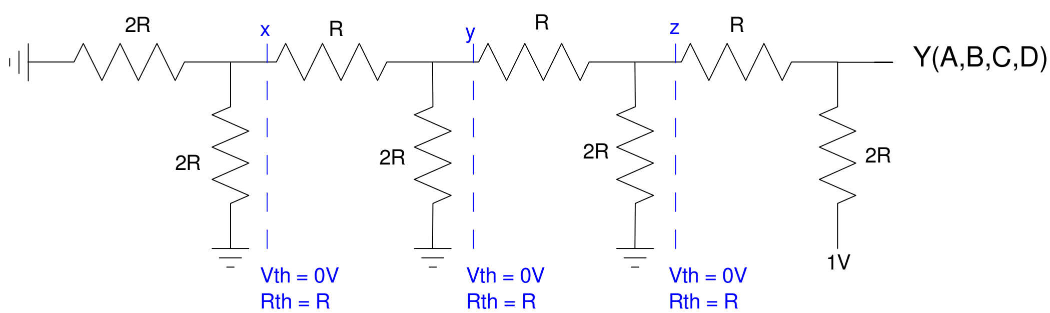

Figure 5: To find the weighting of A, set A=1V and turn off the other inputs

For this circuit, Thevenin equivalents are useful for finding Y.

- At x, the Thevenin equivalent looking left is 0V (not voltage inputs) and Rth = R (2R || 2R = R). Replace the circuit to the left of x with its Thevenin equivalent.
- At y, the Thevenin equivalent looking left is 0V (not voltage inputs) and Rth = R (2R || 2R = R). Replace the circuit to the left of x with its Thevenin equivalent.
- At z, the Thevenin equivalent looking left is 0V (not voltage inputs) and Rth = R (2R || 2R = R). Replace the circuit to the left of x with its Thevenin equivalent.

Finally, at Y you just see a voltage divider:

- 2R goes to 1V
- R + R (2R) goes to ground.

By voltage division, the voltage at Y is 1/2V. This is also the weighting 'a'.

Weighting 'b': To find the weighting for input B,

- Set the voltage at B equal to 1V
- Set the other inputs to 0V

This is shown in Figure 6.

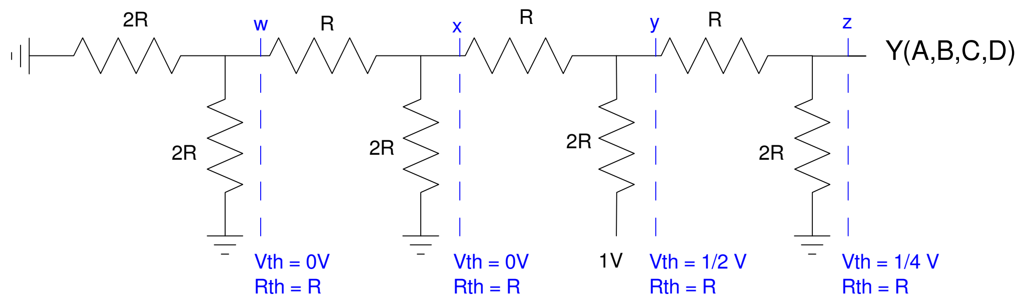

Figure 5: To find the weighting of B, set B=1V and turn off the other inputs

Thinking Thevenin equivalents

- At w looking left, you see a Thevenin equivalent of Rth = R and Vth = 0V
- At x looking left, you see a Thevenin equivalent of Rth = R and Vth = 0V
- At y looking left, you see a Thevenin equivalent of Rth = R and Vth = 1/2V
- Finally, at z looking left, you see a Thevenin equivalent of Rth = R and Vth = 1/4V

The weighting of B is 1/4

Weighting 'c': To find the weighting for input C,

- Set the voltage at C equal to 1V
- Set the other inputs to 0V

This is shown in Figure 7.

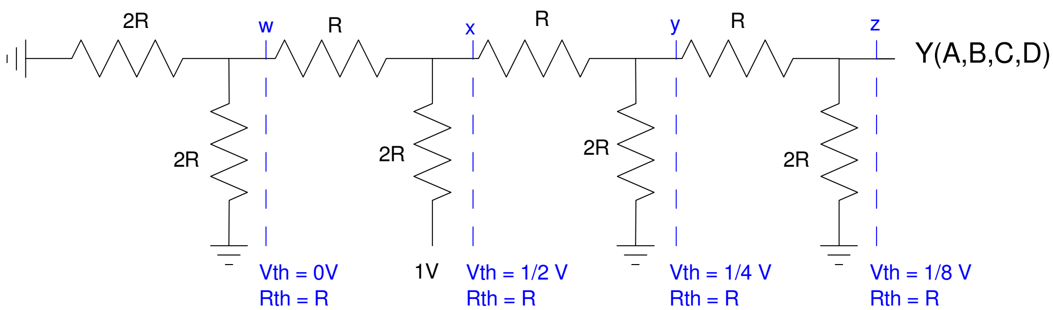

Figure 5: To find the weighting of B, set B=1V and turn off the other inputs

Thinking Thevenin equivalents

- At w looking left, you see a Thevenin equivalent of Rth = R and Vth = 0V
- At x looking left, you see a Thevenin equivalent of Rth = R and Vth = 1/2V
- At y looking left, you see a Thevenin equivalent of Rth = R and Vth = 1/4V

- Finally, at z looking left, you see a Thevenin equivalent of Rth = R and Vth = 1/8V

The weighting of C is 1/8

and so on. The net result is

$$Y = \left(\frac{1}{2}\right)A + \left(\frac{1}{4}\right)B + \left(\frac{1}{8}\right)C + \left(\frac{1}{16}\right)D$$

Note that

- This circuit can be extended indefinitely. Each term has a weighting of 1/2 of the previous term.
- The output is the analog equivalent of the binary signal ABCD. If logic 1 is 5V, then

$$Y = \left(\frac{\text{binary value of ABCD}}{16}\right)5V$$

Hence, this R-2R ladder is a digital-to-analog converter (D/A).

In CircuitLab, you can check these results.

- If you apply a 1Vp sine wave to A and 0V to the rest, the output is a 1/2Vp sine wave at Y.
- If you apply a 1Vp sine wave to B and 0V to the rest, the output is a 1/4Vp sine wave at Y.
- If you apply a 1Vp sine wave to C and 0V to the rest, the output is a 1/8Vp sine wave at Y.
- If you apply a 1Vp sine wave to D and 0V to the rest, the output is a 1/16Vp sine wave at Y.

Figure 6, for example, shows the output when a 1Vp sine wave is applied to B.

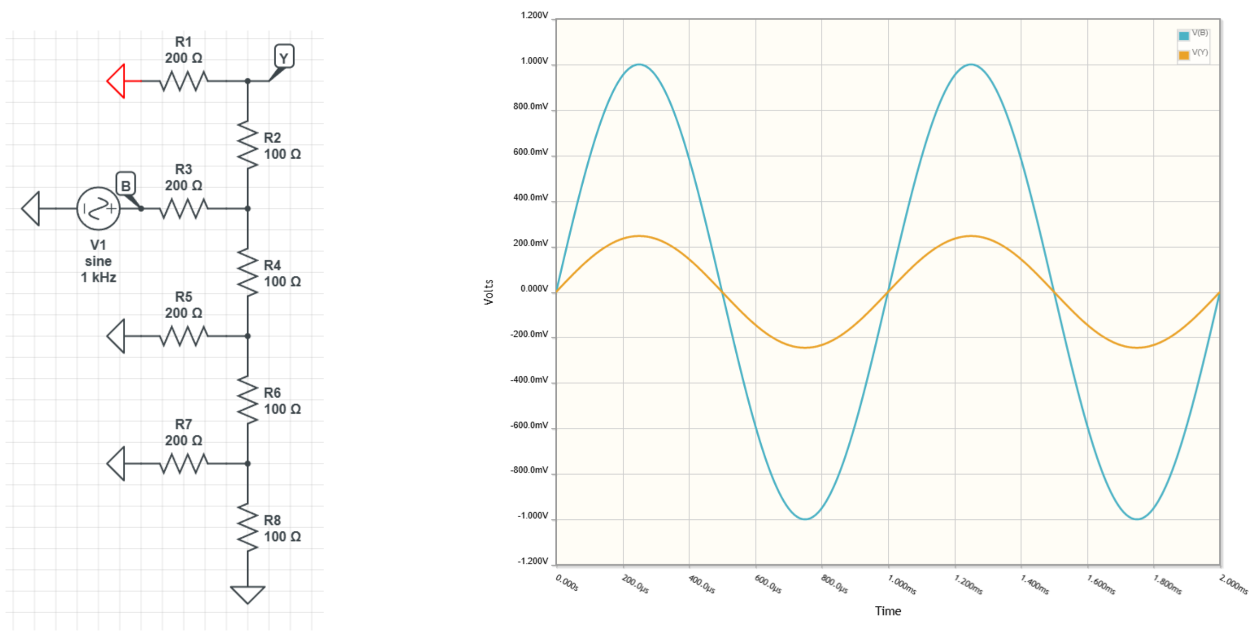

Figure 6: CircuitLab result for a sine wave applied to B. The output (orange) is 1/4th of the input (blue).

Example 3: Weighted Average

Finally, let's design a circuit where the output is the weighted average of three signals. First, assume all three inputs have the same weighting:

$$Y = \left(\frac{A+B+C}{3}\right)$$

By symmetry, this can be implemented using three identical resistors as in Figure 7.

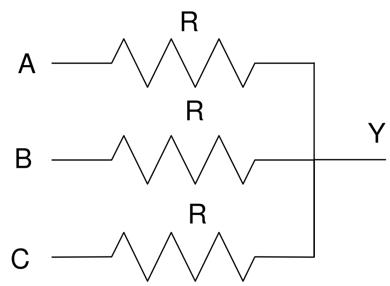

Figure 7: Y is the average of (A, B, C)

You can check this by writing the voltage node equation at Y:

$$\left(\frac{Y-A}{R}\right) + \left(\frac{Y-B}{R}\right) + \left(\frac{Y-C}{R}\right) = 0$$

$$3Y = A + B + C$$

$$Y = \left(\frac{A+B+C}{3}\right)$$

Next, suppose you want the weighting on A to be twice the weighting of B and C.

$$Y = \left(\frac{2A+B+C}{4}\right)$$

One way to do this is to add another resistor to A - giving A double the weight. These two resistors in parallel can be simplified to a single resistor of value R/2 connected to A as shown in Figure 8

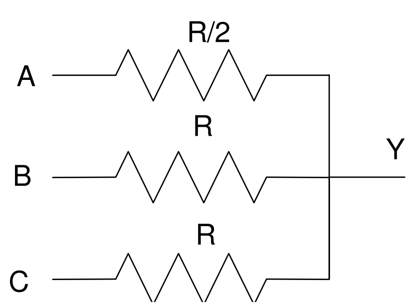

Figure 8: Y is the average of (A, B, C) with A having double the weight of B and C

Finally, design a circuit with arbitrary weightings on A, B and C:

$$Y = \left(\frac{aA+bB+cC}{a+b+c}\right)$$

From the previous solution, this can be done with three resistors in parallel. The resistance of each one is some reference resistance, R, divided by that input's weighting:

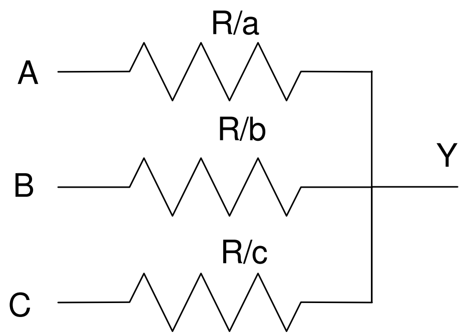

Figure 9: Y is the weighted average of A, B, C

Level Shifting

Using the weighted average of Figure 9, you can design circuits which shift the range of an input's voltage. For example, assume A is an analog signal in the range of -10V .. +10V. Design a circuit to shift this voltage to the range of 0..5V.

Solution: Y is related to A as

$$Y = \frac{1}{4}A + 2.5$$

Assume you have a +5V power supply and a 0V power supply. This can be rewritten as

$$Y = \frac{1}{4}A + \frac{1}{2}(5V) + \frac{1}{4}(0V)$$

or

$$Y = \left(\frac{1\cdot(A)+2(\cdot 5V)+1\cdot(0V)}{4}\right)$$

A circuit which shifts a -10V to +10V signal (A) to a 0..5V signal (Y) is thus

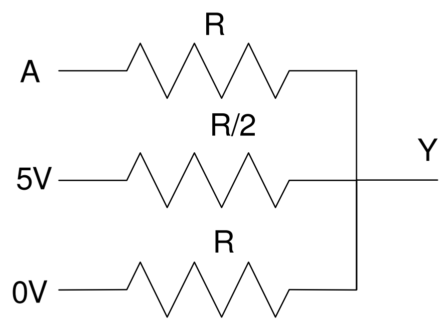

Figure 10: Circuit which converts -10V to +10V at a to 0V to 5V and Y

In CircuitLab, you can check this design by sweeping the voltage at A from -10V to +10V and plotting the voltage at Y as shown in Figure 11. Sure enough, Y goes from 0V to 5V as A goes from -10V to +10V.

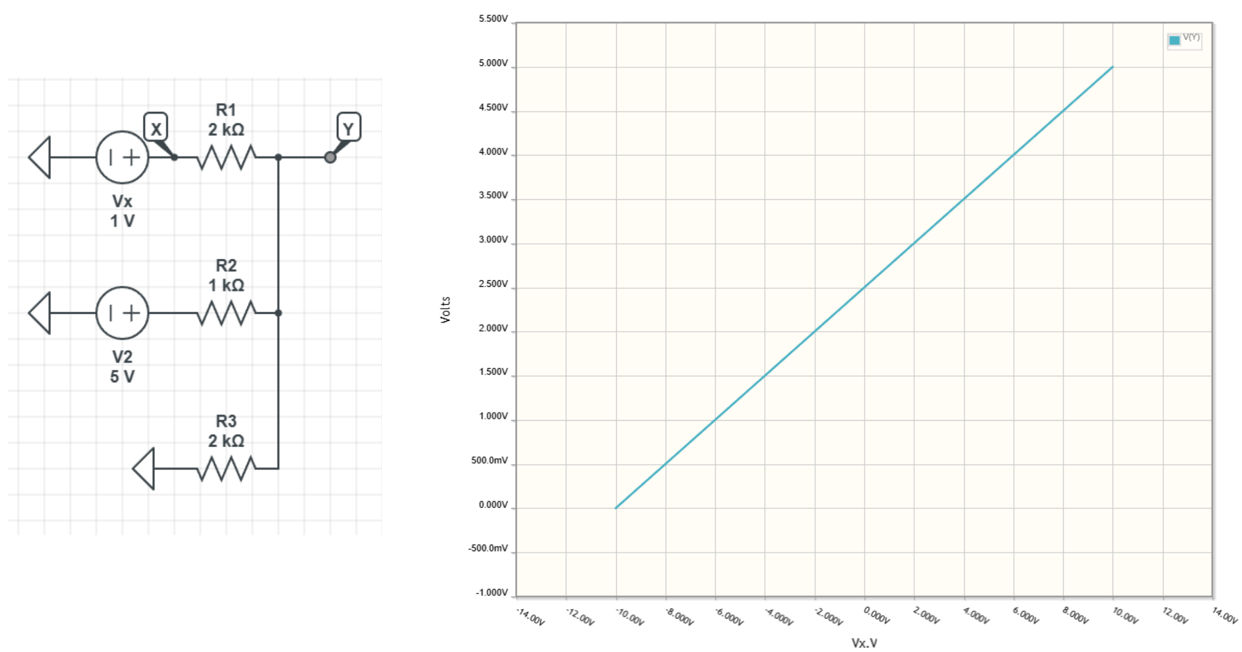

Figure 12: Y goes from 0V to 5V as the input, X, goes from -10V to +10V

Summary

Resistor circuits are linear. This means that

$$f(a+b+c) = f(a) + f(b) + f(c)$$

This ability to take a circuit with several inputs and analyzing them separately is called *superposition*. Superposition can simplify circuit analysis, allowing you to analyze a bunch of simpler circuits individually to find the overall answer.