Thevenin Equivalents & Max Power Transfer

Background

Given any linear circuit with an output, the VI characteristics at the output follows a load line. As far as the load is concerned, it doesn't know or care how you create that load line: all circuits with the same load line look the same as far as the load is concerned.

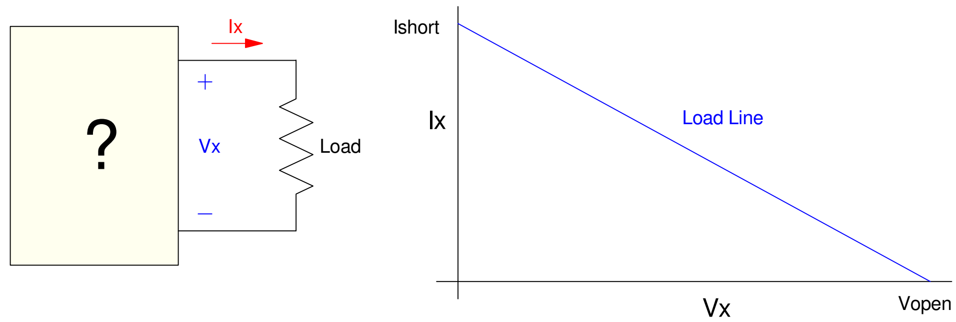

Figure 1: For a linear circuit, the V/I characteristic at the output will follow a load line

The simplest circuits which produce a given load line are the Thevenin and Norton equivalents.

In the previous lecture, techniques to find the Thevenin and Norton equivalent circuits for circuits with independent sources were covered. This lecture will cover

- Finding Thevenin equivalents for circuits with dependent sources, and
- Find the load that results in the maximum power being delivered to the load.

Types of Dependent Sources

Dependent sources are either voltage sources or current sources whose value depend upon another circuit parameter. Examples of dependent sources include

- Current-controlled current sources (BJT transistor)
- Voltage-controlled voltage sources (op-amp amplifier), and
- Voltage-controlled current sources (Mosfet)

An upcoming lecture will cover op-amps. The other devices are covered in Electronics. For this class, just treat these as just another source.

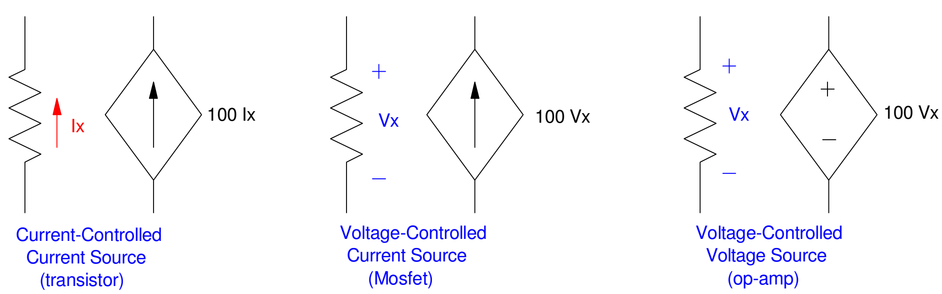

Figure 2: Types of dependent sources

Case 1: Current-Controlled Current Source

Let's start with a current-controlled current source. An example of such a device is a bipolar junction transistor - something covered in Electronics. For this class, just treat it like any other circuit element.

Suppose you have a circuit as shown in Figure 3. For this circuit, find the Thevenin equivalent at the output.

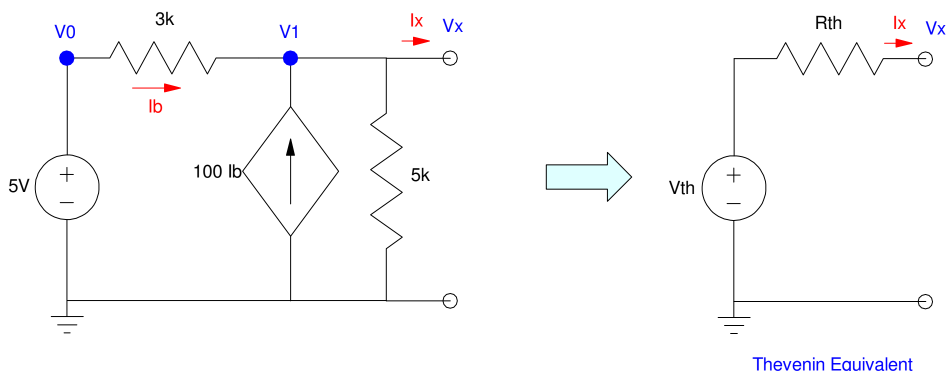

Figure 3: Find the Thevenin equivalent

To find the Thevenin equivalent, two of the following need to be found:

- Open-circuit voltage,
- Short-circuit current,
- Thevenin resistance

Open-Circuit Voltage: To find the open-circuit voltage, simply solve the above circuit as shown (i.e. with no load connected to Vx). Writing the voltage node equations at V1:

$$I_b = \left(\frac{5-V_1}{3k}\right)$$

$$-I_b - 100I_b + \left(\frac{V_1}{5k}\right) = 0$$

Substitute and solve:

$$V_1 = \left(\frac{\left(\frac{101}{3k}\right)}{\left(\frac{101}{3k}\right)+\left(\frac{1}{5k}\right)}\right)5V = 4.9705V$$

This gives Vth

V(open) = Vth = 4.9705V

Short-Circuit Current: To fidn the short-circuit current, short Vx to ground as shown in Figure 4 and compute the resulting current draw.

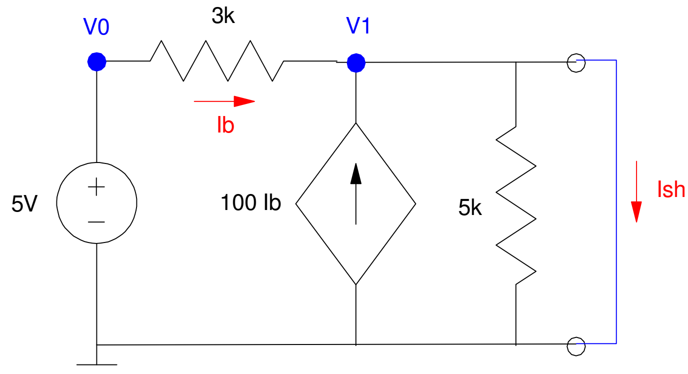

Figure 4: To find the short-circuit current, short the output to ground and compute the current draw

By observation, V1 = 0V. This results in

$$I_b = \left(\frac{5V-0V}{3k}\right) = 1.667mA$$

Writing the node equation at V1

$$-I_b - 100I_b + \left(\frac{V_1}{5k}\right) + I_{short} = 0$$

This gives the short-circuit current

$$I_{short} = 168.33mA$$

Thevenin Resistance: The Thevenin resistance can be found two ways. First, if you know the open-circuit voltage and the short-circuit current, then you know Rth

$$R_{th} = \frac{V_{open}}{I_{short}} = 29.53\Omega$$

A second option is to turn off the independent sources then compute the resistance looking in.

In the last lecture, the resistance looking in was fairly easy to calculate. With the dependent source, it's no longer obvious what the net resistance is. In such a case, one trick you can use is to apply a 1V test voltage at the output as shown in Figure 5. If you find the resulting current draw, you can compute Rth as

$$R_{th} = \frac{V_x}{I_x}$$

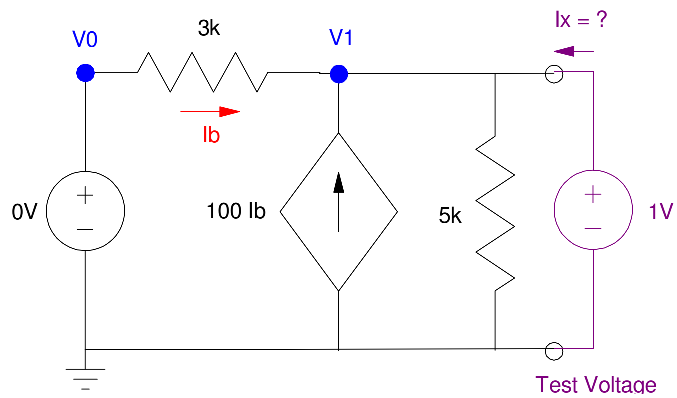

Figure 5: A test voltage is added to the output. The Thevenin resistance is then Vx/Ix

For this circuit,

$$I_b = \left(\frac{0V-1V}{3k}\right) = -333.3\mu A$$

From conservation of current at V1

$$I_x = -I_b - 100 I_b + \left(\frac{1V}{5k}\right)$$

This results in

$$I_x = 33.87mA$$

and

$$R_{th} = \frac{V_x}{I_x} = \frac{1V}{33.87mA} = 29.53\Omega$$

This is a common trick when dealing with dependent sources: applying a test voltage and then compute the resulting current draw.

Case 2: Voltage-Controlled Voltage Source

Next, consider a voltage-controlled voltage source as shown in Figure 6. This is actually the circuit model of an op-amp that we'll cover in a future lecture. For now, just treat it like any other circuit.

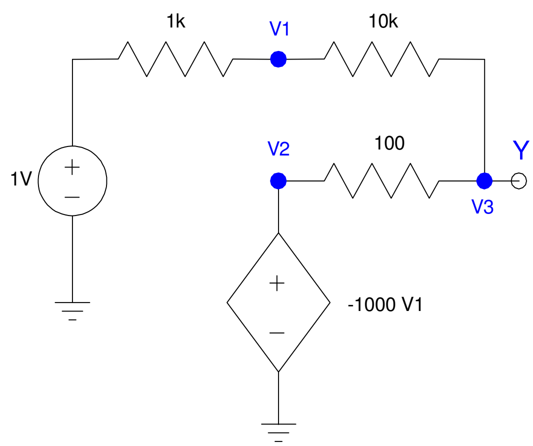

Figure 6: Find the Thevenin equivalent of circuit containing a voltage-controleld voltage-source

The procedure will the same as before:

- Find the open-circuit voltage
- Find the short-circuit current, and
- Find the Thevenin resistance.

Open-Circuit Voltage: For this, leave the output, Y, without any load. Compute the voltage at Y using circuit techniques.

Writing the three voltage node equations

$$V_2 = -1000V_1$$

$$\left(\frac{V_1 - 1}{1k}\right) + \left(\frac{V_1 - V_3}{10k}\right) = 0$$

$$\left(\frac{V_3 - V_1}{10k}\right) + \left(\frac{V_3 - V_2}{100}\right) = 0$$

Next, group terms, place in matrix form, then find the voltages. Skipping these steps, the result is

```
A = [1000,1,0;11,0,-1;-1,-100,101];
B = [0;10;0];
V =  inv(A)*B

V1    0.0100
V2   -9.9891
V3   -9.8901      =  Vth
```

Short-Circuit Current: For this, short Y to ground as shown in Figure 7 and compute the current through the short.

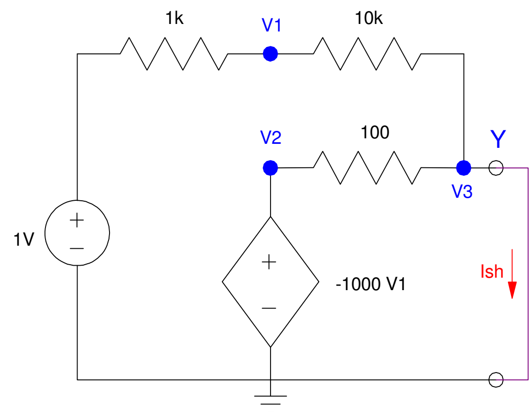

Figure 7: To find the short-circuit current, short the output to ground and compute the current draw

From voltage division, V1 is

$$\left(\frac{V_1 - 1}{1k}\right) + \left(\frac{V_1 - 0}{10k}\right) = 0$$

$$V_1 = 0.90909V$$

The short-circuit current is then

$$I_{sh} = \left(\frac{1V}{11k}\right) - \left(\frac{1000V_1}{100}\right)$$

$$I_{sh} = -9.0908A$$

Thevenin Resistance: Once again, there are two ways to get this. First, if you know the open-circuit voltage and the short-circuit current, then

$$R_{th} = \frac{V_{open}}{I_{short}} = 1.088\Omega$$

Another option is to

- Turn off voltage sources
- Apply a 1V test voltage
- Compute the current

First, turn off the voltage sources and apply a 1V test signal to the output. This is shown in Figure 8.

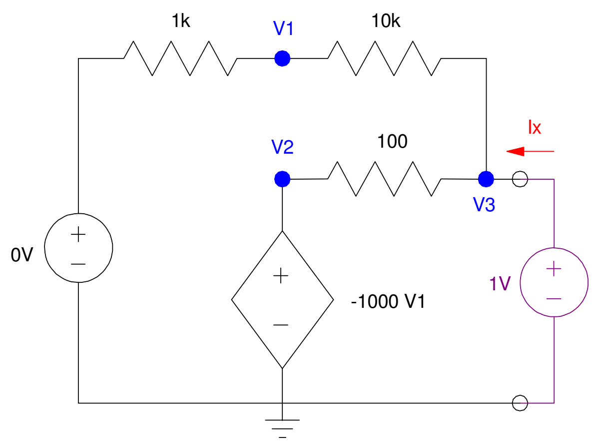

Figure 8: Turn off independent sources and apply a 1V test signal at the output

The resulting current, Ix, can be found as

$$V_1 = \left(\frac{1k}{1k+10k}\right) 1V = 90.91mV$$

$$V_2 = -1000V_1 = -90.91V$$

$$I_x = \left(\frac{1V}{11k}\right) + \left(\frac{1V-V_2}{100}\right) = 919.2mA$$

Rth is thus

$$R_{th} = \frac{V_{in}}{I_{in}} = \frac{1V}{919.2mA} = 1.088\Omega$$

Max Power Transfer

Switching gears a little, consider the following problem. Suppose you have a circuit with an output as in Figure 9.

- What load resiances maximizes the power to the load?
- How much power *can* you deliver to the load?

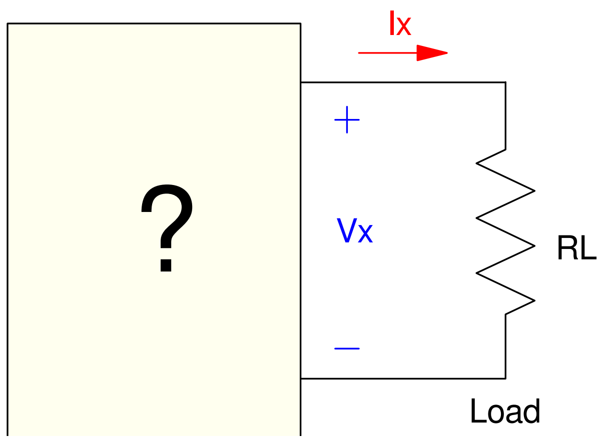

Figure 9: What value of RL maximizes the power to the load?

This is another example of where Thevenin equivalents are useful. Assuming you're dealing with a linear circuit, that circuit can be replaced with its Thevenin equivalent. Let's do that. The problem then becomes that of finding RL to maximize the power at the load in Figure 10.

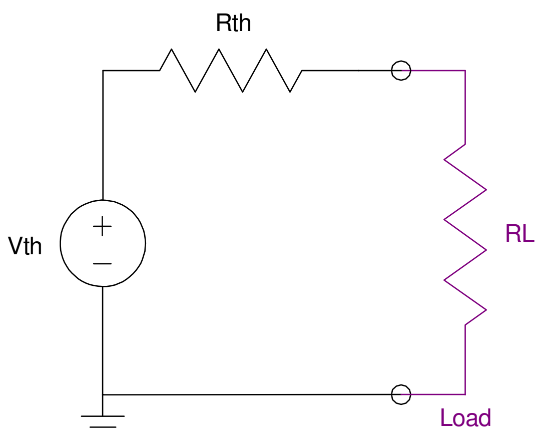

Figure 10: Equivalent problem: find RL to maximize the power to the load

The solution depends upon which resistor you are allowed to vary: Rth or RL.

Case 1: Rth is fixed and RL can vary.

In this case, compute the power to the load as a function of RL. The current to the load is

$$I = \left(\frac{V_{th}}{R_{th}+R_L} \right)$$

The power to the load is

$$P = I^2 R_L$$

$$P = \left(\frac{R_L}{(R_{th}+R_L)^2} \right) V_{th}^2$$

The maximum will be when the derivative is zero

$$\frac{d}{dR_L}\left(\frac{R_L}{(R_{th}+R_L)^2} \right) = 0$$

Simplifying

$$(R_L + R_{th})(R_{th} - R_L) = 0$$

This gives two solutions:

- RL = Rth maximum power to the load
- RL = -Rth minimum power to the load

Maximum power is delivered to the load when RL = Rth

To see this, let's assume

- Vth = 12V
- Rth = 2 Ohms

In Matlab, the power to the load can be found for 0 < RL < 10 Ohms:

```
>> Vth = 12;
>> Rth = 2;
>> RL = [0:0.05:10]';
>> I = Vth ./ (Rth + RL);
>> V = RL ./ (Rth + RL) * Vth;
>> P = V .* I;
>> plot(RL, P);
```

The resulting power vs. RL is shown in Figure 11. Note that

- The maximum occurs when RL = Rth = 2 Ohms
- The efficiency at this point is only 50%

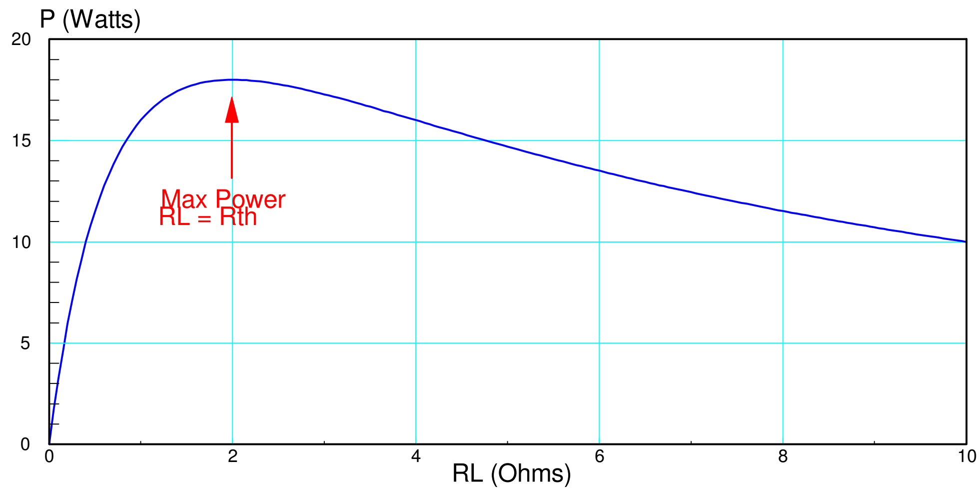

Figure 11: Power to the load vs. RL. Note that the maximum is when RL = Rth = 2 Ohms

Case 2: RL is fixed and Rth can vary.

If you can vary Rth, the power to the load becomes

$$P = \left(\frac{R_L}{(R_{th}+R_L)^2}\right) V_{th}^2$$

To find the maximum, take the derivative and set it equal to zero

$$\frac{dP}{dR_{th}} = 0 = \left(\frac{-2(R_{th}+R_L)R_L}{(R_{th}+R_L)^3}\right) V_{th}^2$$

This has a solution of

$$R_{th} = -R_L$$

At that point, the power to the load is infinite. If negative resistances are not possible, Rth should be as small as possible. Zero ideally.

You can see this in Figure 12. Assuming

- Vth = 12V
- RL = 2 Ohms

the power to the load can be found using Matlab as

```
>> Vth = 12;
>> RL = 2;
>> Rth = [0:0.05:10]';
>> I = Vth ./ (Rth + RL);
>> V = RL ./ (Rth + RL) * Vth;
>> P = V .* I;
>> plot(Rth, P);
```

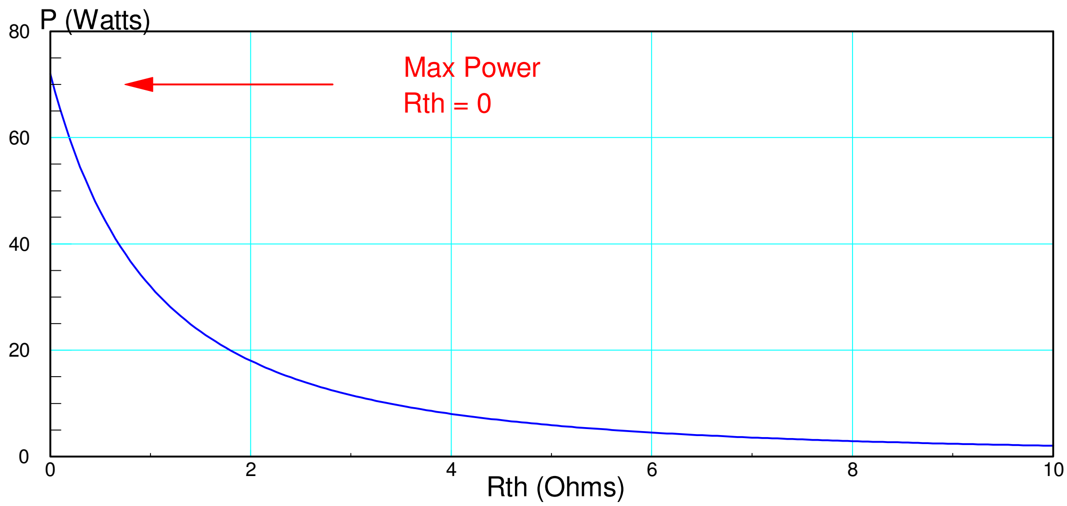

Figure 12: Power to the load as Rth varies. Max power is when Rth = 0

Case 3: Find RL to maximize efficiency.

Efficiency is the power to the load divided by the total power delivered by the source.

$$eff = \left(\frac{P_{Load}}{P_{total}}\right)$$

or

$$eff = \frac{I^2 R_L}{I^2 (R_{th} + R_L)} = \left(\frac{R_L}{R_L + R_{th}}\right)$$

Maximum efficiency is when RL is infinity. At this point

- R_L = infinity
- Efficiency = 100%
- Power = 0

That's one of the problems of delivering power

- For high efficiency, you want $R_L >> R_{th}$
- For maximum power, you want $R_L = R_{th}$

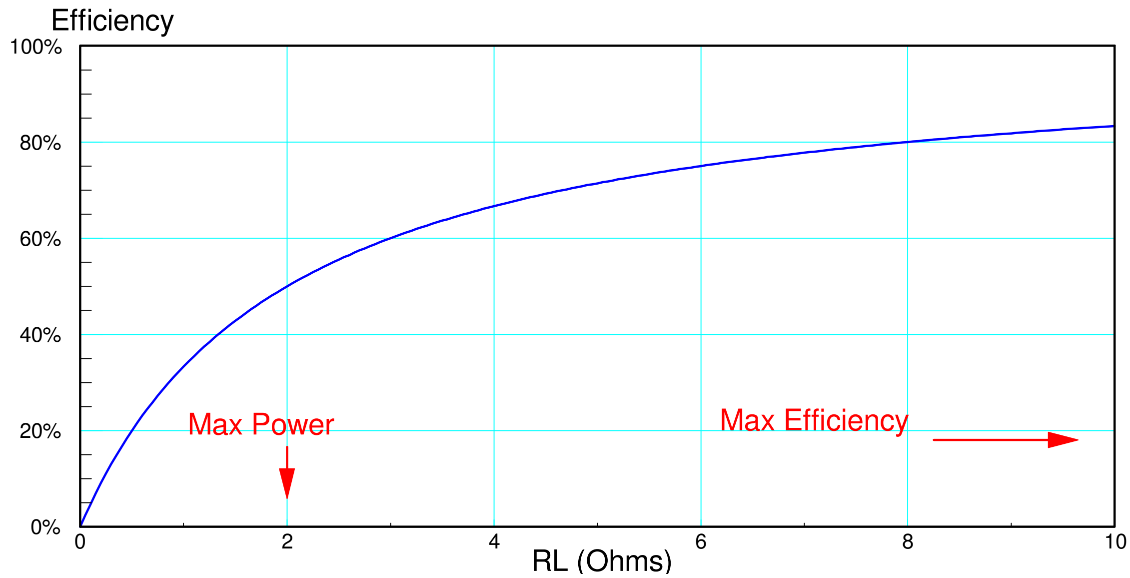

Figure 13: Efficiency vs. RL. Large RL increase efficiency but reduce the power to the load

Summary

Wen circuits have a dependent source, computations for Rth can get tricky. One technique to find Rth is to apply a 1V test voltage at the output. The resulting current draw, Ix, determines Rth as

$$R_{th} = \frac{1V}{I_x}$$

Thevenin equivalents are also useful when determining the maximum power you can deliver to a load. This maximum occurs when RL = Rth. At this point, however, the efficiency of the system is only 50%.