Fourier Transform

Solving differential equations when the input is periodic

ECE 111 - Week #15

Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

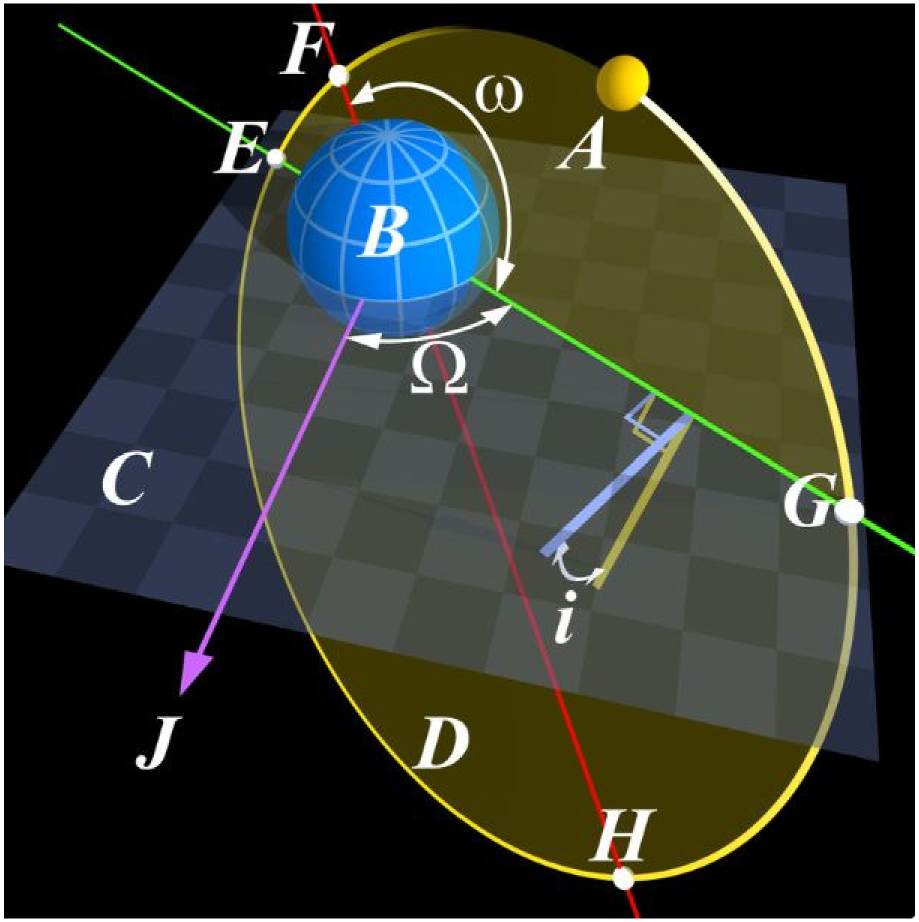
Solutions to Differential Equations

Differential equations are important.

- They pretty much describe everything
- Circuits, weather, economics,
- Ballistics, celestial mechanics,
- Populations, disease spread, pollution, etc

Likewise, being able to solve differential equations is also important

- Topic for Calculus III, Calc IV
- Topic for Circuits II, Signals & Systems
- Solution used throughout the ECE curriculum



Solution for Sinusoidal Inputs

- Calculus III method

Assume you have a differential equaton with a sinusoidal input

$$y'' + 3y' + 8y = 20x$$

$$x(t) = 5\sin(6t)$$

The method used in Calculus is to guess the answer

$$y(t) = a\cos(6t) + b\sin(6t)$$

Take derivatives and substitute

$$y' = -6a\sin(6t) + 6b\cos(6t)$$

$$y'' = -36a\cos(6t) - 36b\sin(6t)$$

This lets you solve 2 equtions for 2 unknowns

- One for cosine terms
- One for sine terms

Solution using Phasors

In Circuits II, phasors are used

- Same topic we covered last week

Start with the problem

$$y'' + 3y' + 8y = 20x$$

$$x(t) = 5\sin(6t)$$

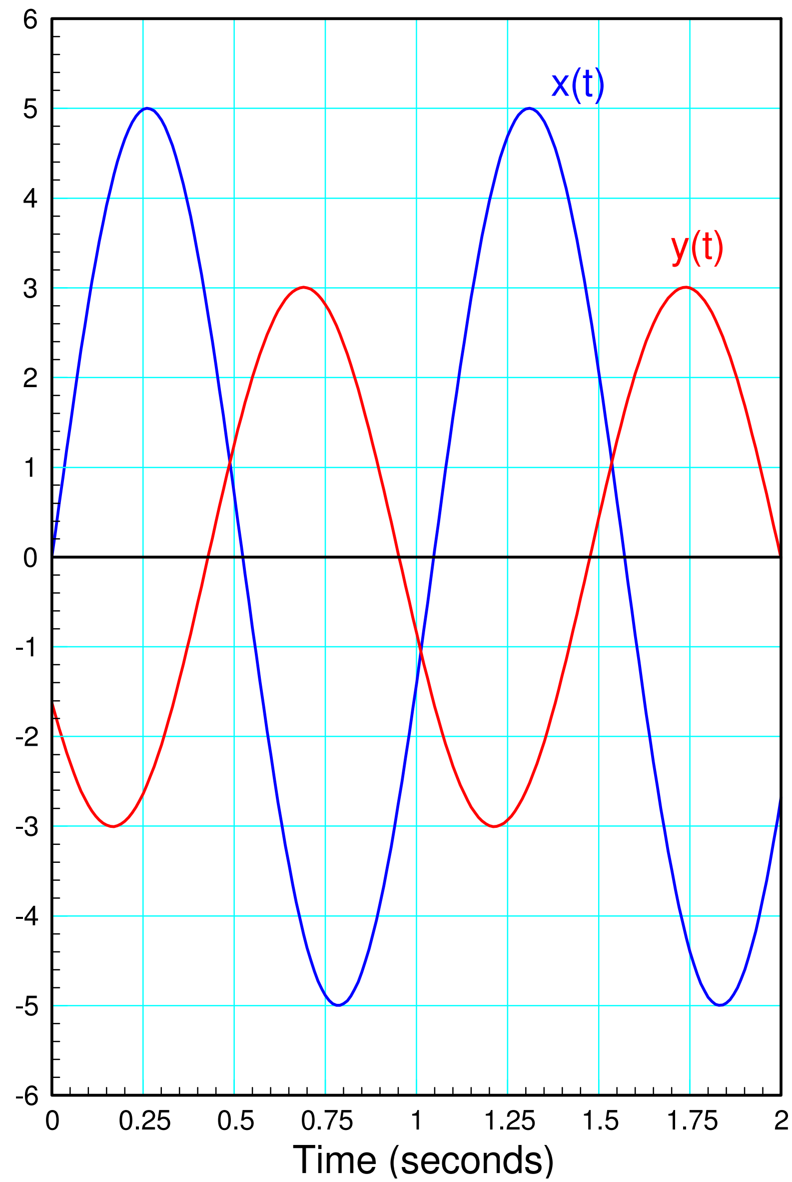
Rewrite in phasor form and solve using complex numbers

$$Y = \left(\frac{20}{s^2+3s+8}\right)_{s=j6} \cdot (0 - j5)$$

$$Y = -1.6245 + j2.5271$$

meaning

$$y(t) = -1.6245\cos(6t) - 2.5271\sin(6t)$$



What if it isn't a sine wave?

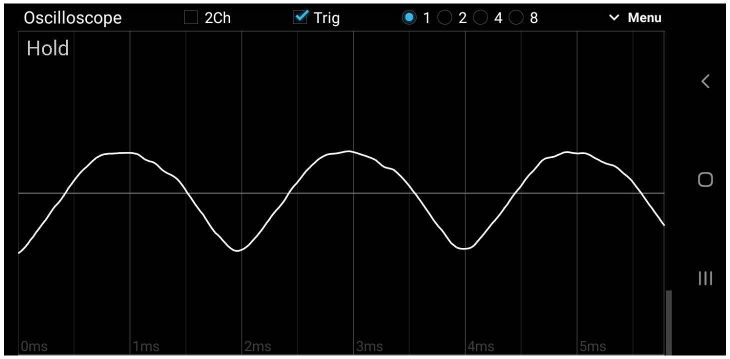
- https://midi.city/
- Oscilloscope app on your cell phone

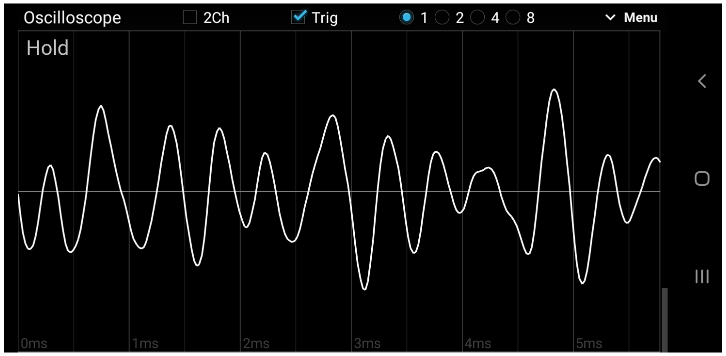
What if you have

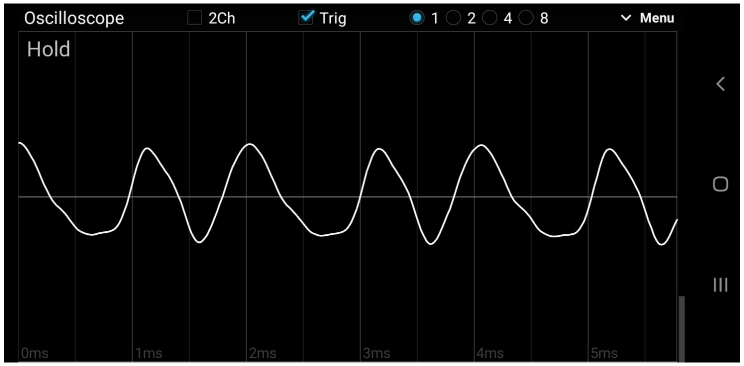
- A differential equation with
- A periodic input
- That is *not* a sine wave

Example Note A4 (500Hz):

- Piano
- Tubular Bells
- Cello







Solution: This Lecture

- Change the problem.

Use the property of linearity:

$$f(a+b)=f(a)+f(b)$$

Approximate the input as the sum of sine waves

Once you have sine waves, we know what to do

- Use phasor analysis for each sine wave
- Add them up to get the total response

This technique uses *Fourier Transforms*

- Named for Joseph Fourier
- Topic covered in Circuits II, Signals and Systems
- Here, we will use Matlab

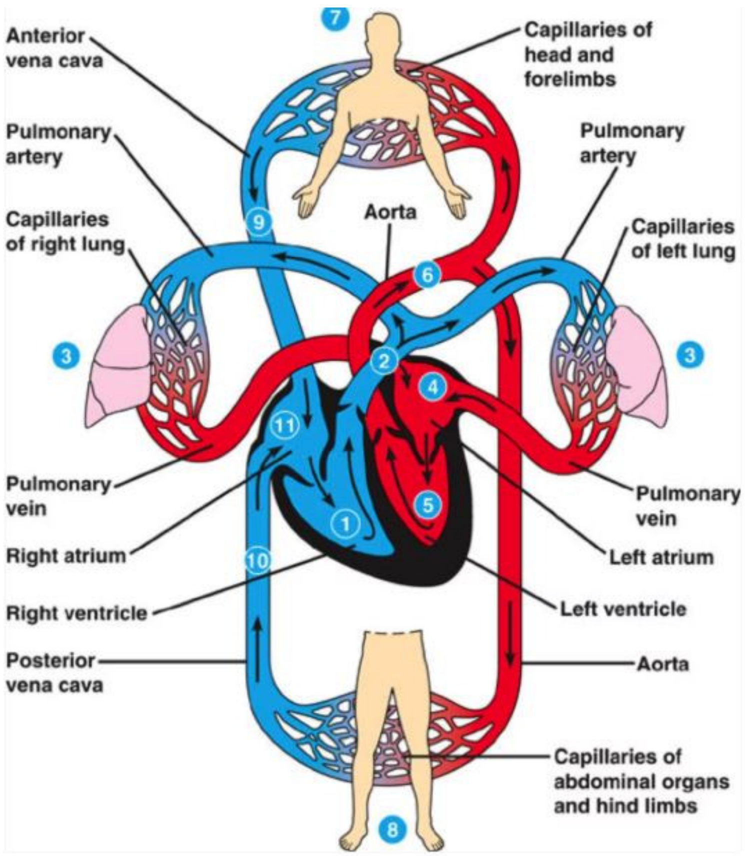
Importance of Fourier Transforms

With this technique, you can solve

- Any differential equation with
- Any periodic input

Example used in this lecture:

- Heart pumping blood to the body
- Model the cardiovascular system with a differential equation
- Model the aortic flow as a periodic waveform
  - Input X
- Determine the resulting aortic pressure
  - Output Y



Step 1: Model the Cardiovascular System

- Find the Transfer Function for a Winkessel Model

Winkessel Model

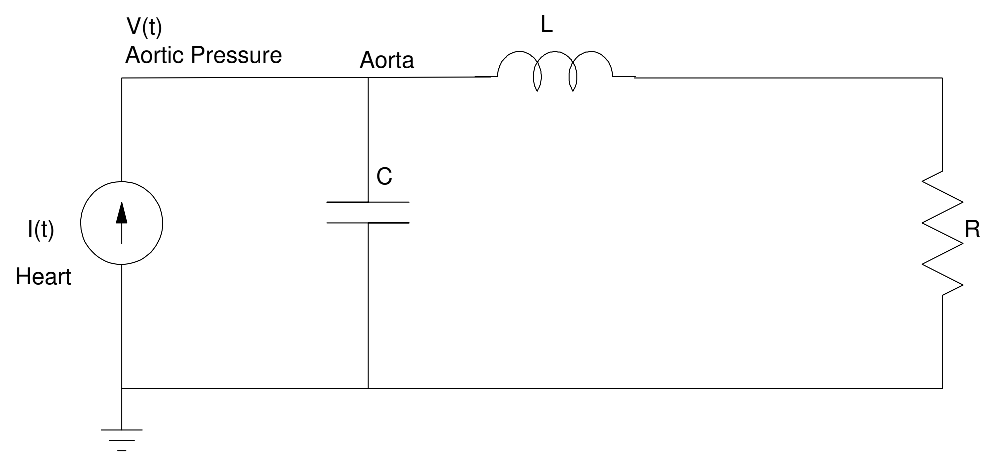
- Models the cardiovascular system
- Given I(t), determine the V(t)
- Given aortic flow (AOF), determine aortic pressure (AOP)

Phasors and s-Notation

$$R \to R$$

$$L \to j\omega L = Ls$$

$$C \to \frac{1}{j\omega C} = \frac{1}{Cs}$$



The resistance of the above circuit is then

$$Z = \left(\frac{1}{1/Cs} + \frac{1}{Ls+R}\right)^{-1} = \left(\frac{Ls+R}{CLs^2+CRs+1}\right)$$

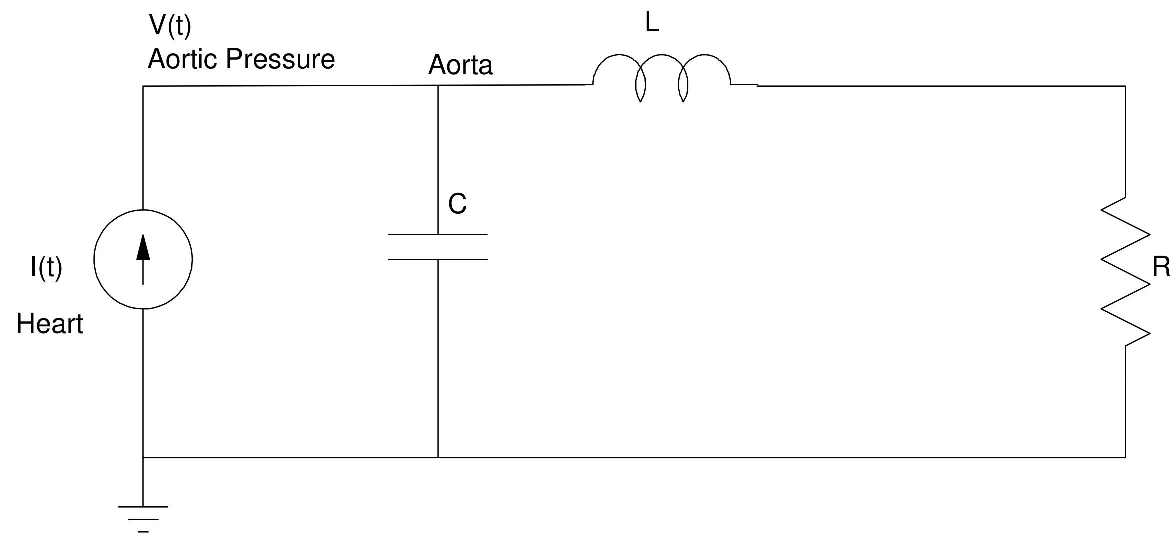
Winkessel Model:

Assume

- R = 1
- L = 0.1
- C = 1

you get

$$V = \left(\frac{s+10}{s^2+10s+10}\right) I$$



Problem:

- Assume I(t) is periodic
- How do you find V(t)?

Case 1: Sinusoidal Input

Assume

$$I(t) = 3\sin(10t)$$

This is a phasor problem.

$$I(t) = 0 - j3$$

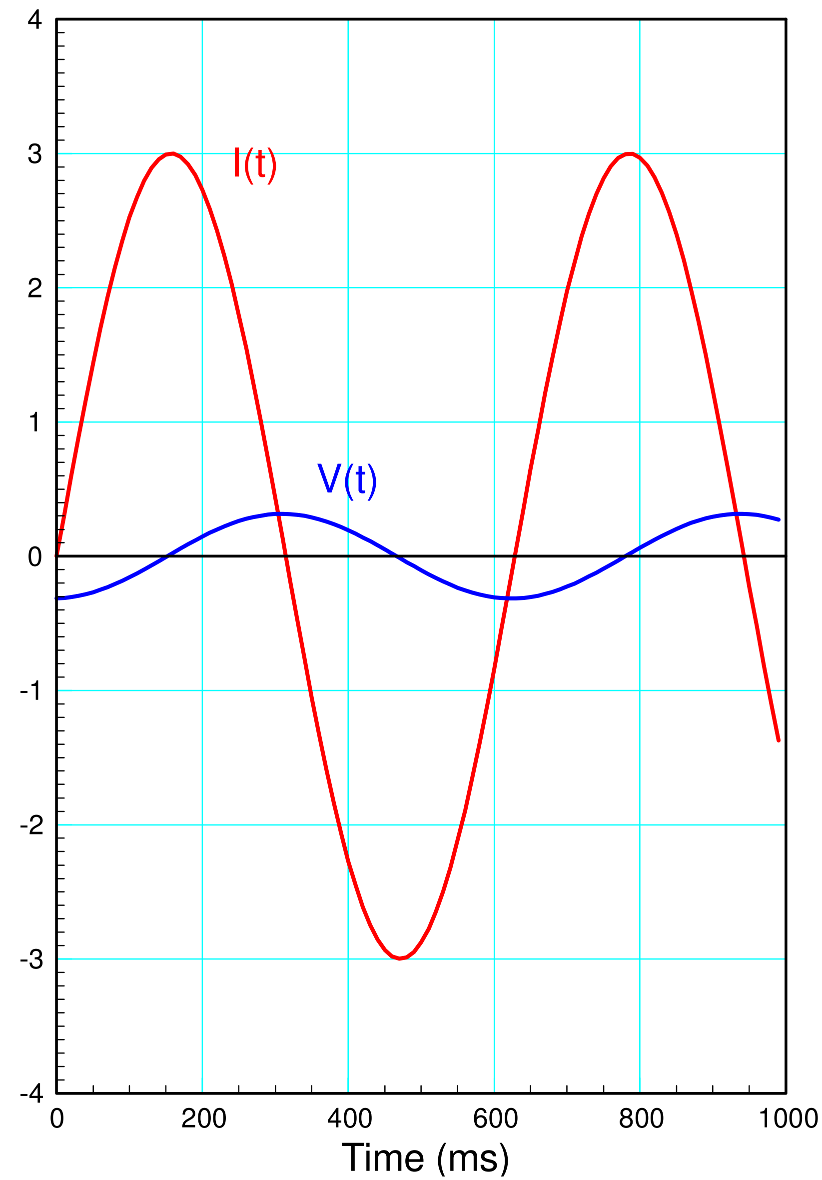
$$s = j10$$

$$V = \left.\left(\frac{s+10}{s^2+10s+10}\right)\right|_{s=j10} \cdot (0 - j3)$$

$$V = -0.3149 - 0.0166i$$

meaning

$$v(t) = -0.3149\cos(10t) + 0.0166\sin(10t)$$



Case 2: Input is Periodic but Not a Sinusoid

- What do you do when i(t) is *not* a sinusoid?

Find V(t) when

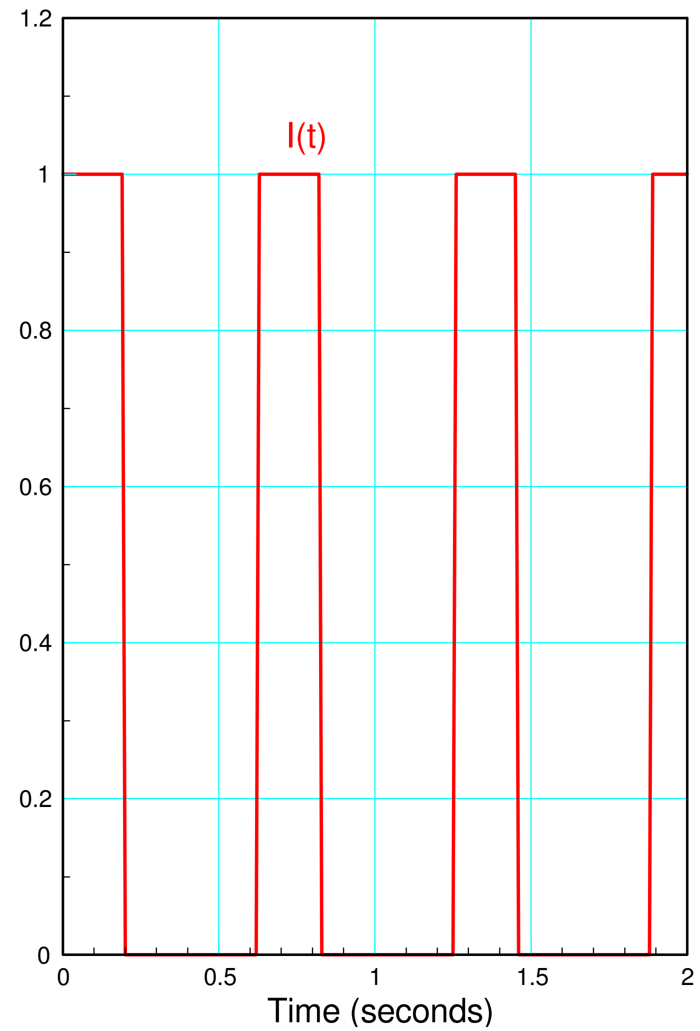
$$V = \left(\frac{s+10}{s^2+10s+10}\right) I$$

I(t) is periodic every $\frac{2\pi}{10}$ seconds

$$I\left(t + \frac{2\pi}{10}\right) = I(t)$$

and

$$I(t) = \begin{cases} 1 & 0 < t < 0.2 \\ 0 & otherwise \end{cases}$$



Change the Problem

Use curve fitting to approximate x(t) with a series expansion

Example: Power Series

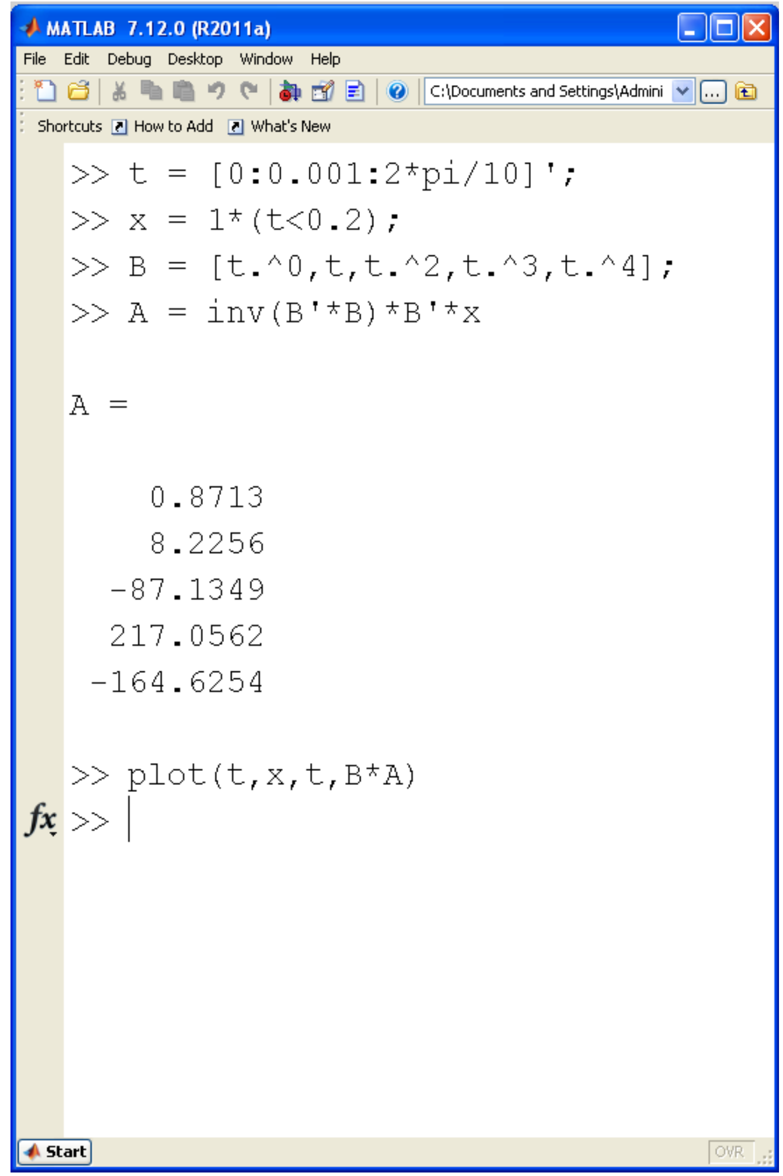
$$x(t) \approx a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \ldots$$

Let the Basis function be

$$B = [1, t, t^2, t^3, t^4]$$

Find the constants using Matlab

```
a0 =       0.8713
a1 =       8.2256
a2 =     -87.1349
a3 =     217.0562
a4 =    -164.6254
```



Problem with a Power Series Solution...

- It doesn't help solve the problem

Phasors allow us to solve differential equations with sinusoidal inputs

- They don't help when the input is a power series...

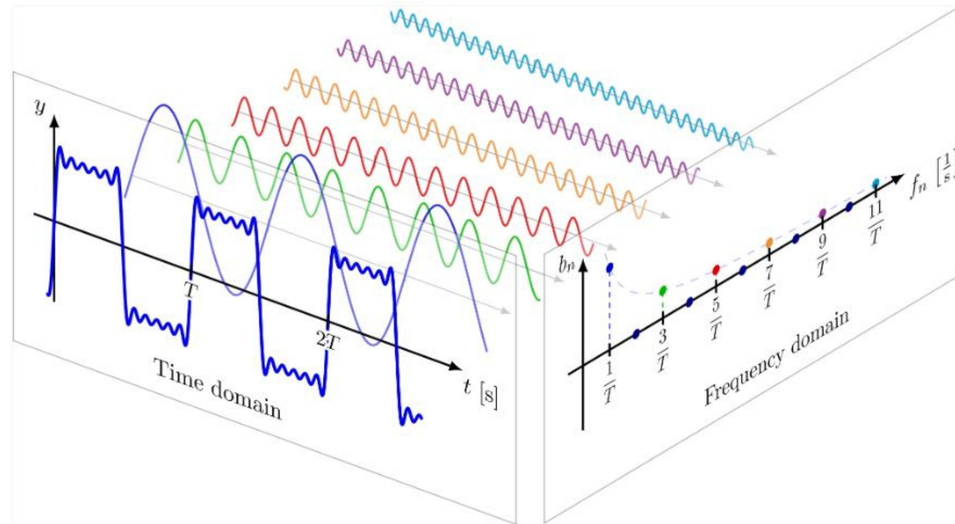
Change the Problem (take 2)

Instead, choose a basis which *is* made up of sine waves

$$x(t) = a_0 + a_1 \cos(\omega_0 t) + b_1 \sin(\omega_0 t)$$
$$+ a_2 \cos(2\omega_0 t) + b_2 \sin(2\omega_0 t) + ..$$

This will result in something that *is* useful

- Sine waves



Expressing x(t) in this way is called *The Fourier Transform of x(t)*

Fourier Transform

Assume

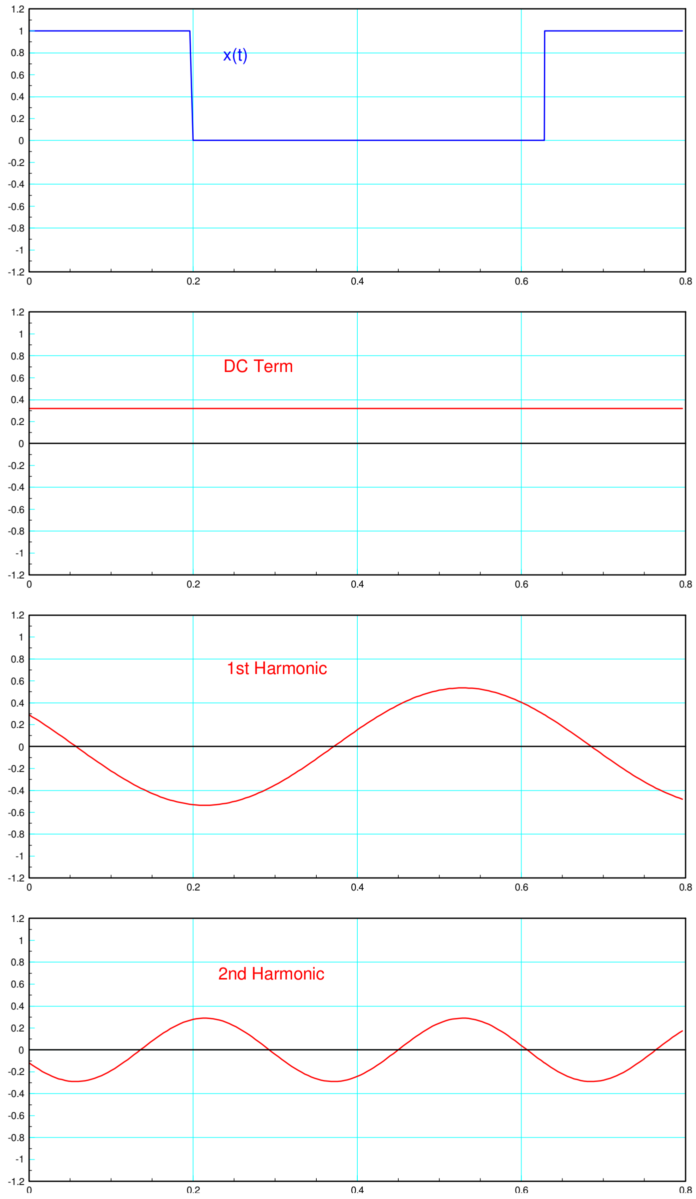
$$x(t) = x(t+T)$$

then

$$x(t) = a_0 + \sum a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

where

$$\omega_0 = \frac{2\pi}{T}$$



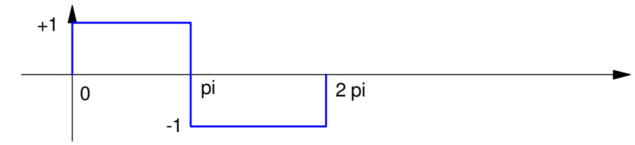
Translation:

- A signal made up of signals which are periodic in time T is also periodic in time T (duh)
- A signal which is periodic in time T is made up of harmonics

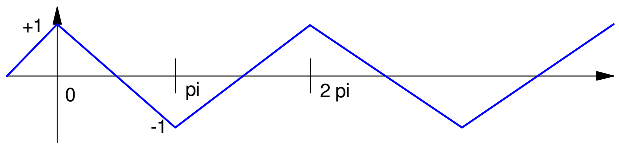
Problem: How to go right to left

- Add up a bunch of periodic signals and the result is periodic
- Adjust the amplitudes of the sine & cosine terms to get different waveforms.

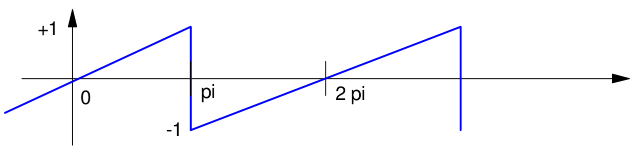
Table of Fourier Transforms (CRC Handbook of Mathematics)



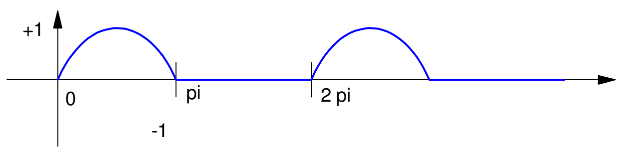
$$\frac{4}{\pi} \sum_{\text{n odd}} \left(\frac{1}{n}\right) \sin(nt)$$



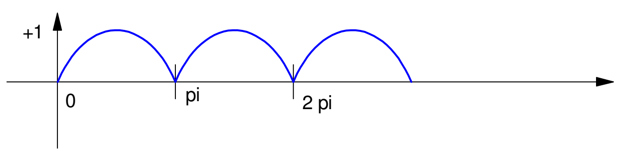
$$\frac{8}{\pi^2} \sum_{\text{n odd}} \left(\frac{1}{n^2}\right) \cos(nt)$$



$$\frac{2}{\pi} \sum_{n} (-1)^{n-1} \left(\frac{1}{n}\right) \sin(nt)$$



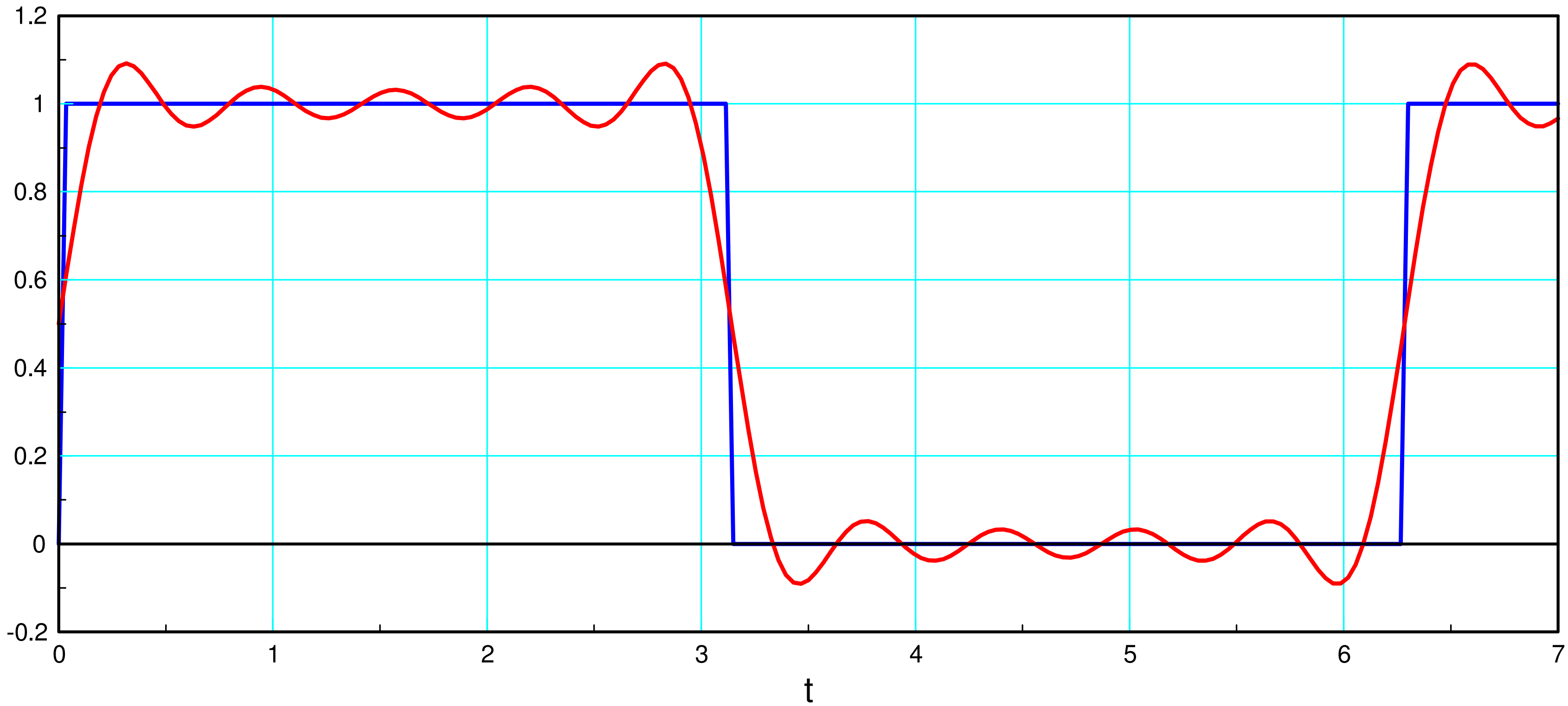
$$\frac{1}{\pi} + \frac{1}{2}\sin(x) - \frac{2}{\pi} \sum_{\text{n even}} \frac{1}{n^2-1} \cos(nt)$$



$$\frac{2}{\pi} - \frac{4}{\pi} \sum_{\text{n even}} \frac{1}{n^2-1} \cos(nt)$$

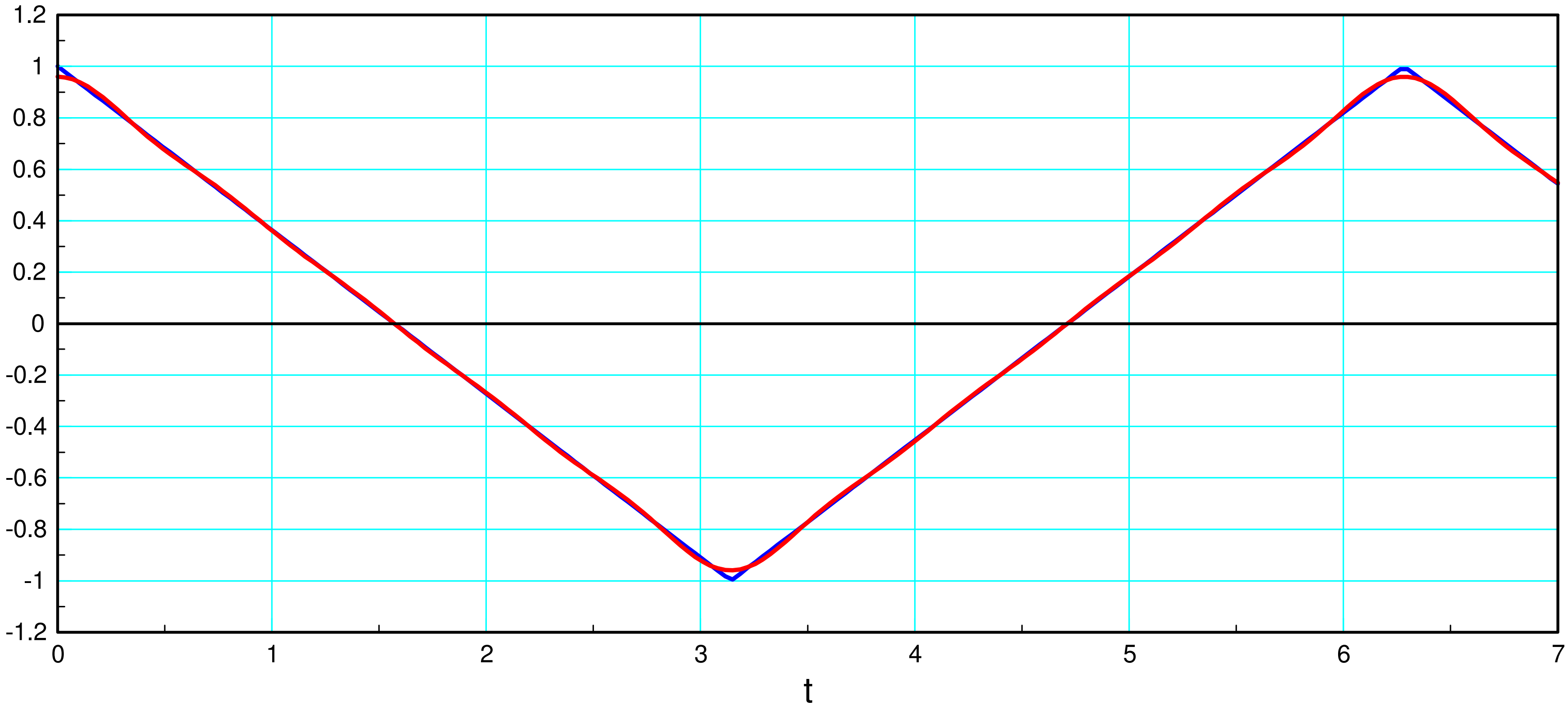
Example: Square Wave

- $x(t) = 0.5 + \sum_{\text{n odd}} \frac{2}{\pi n} \sin(nt)$
- n = 9 (red) & infinity (blue)



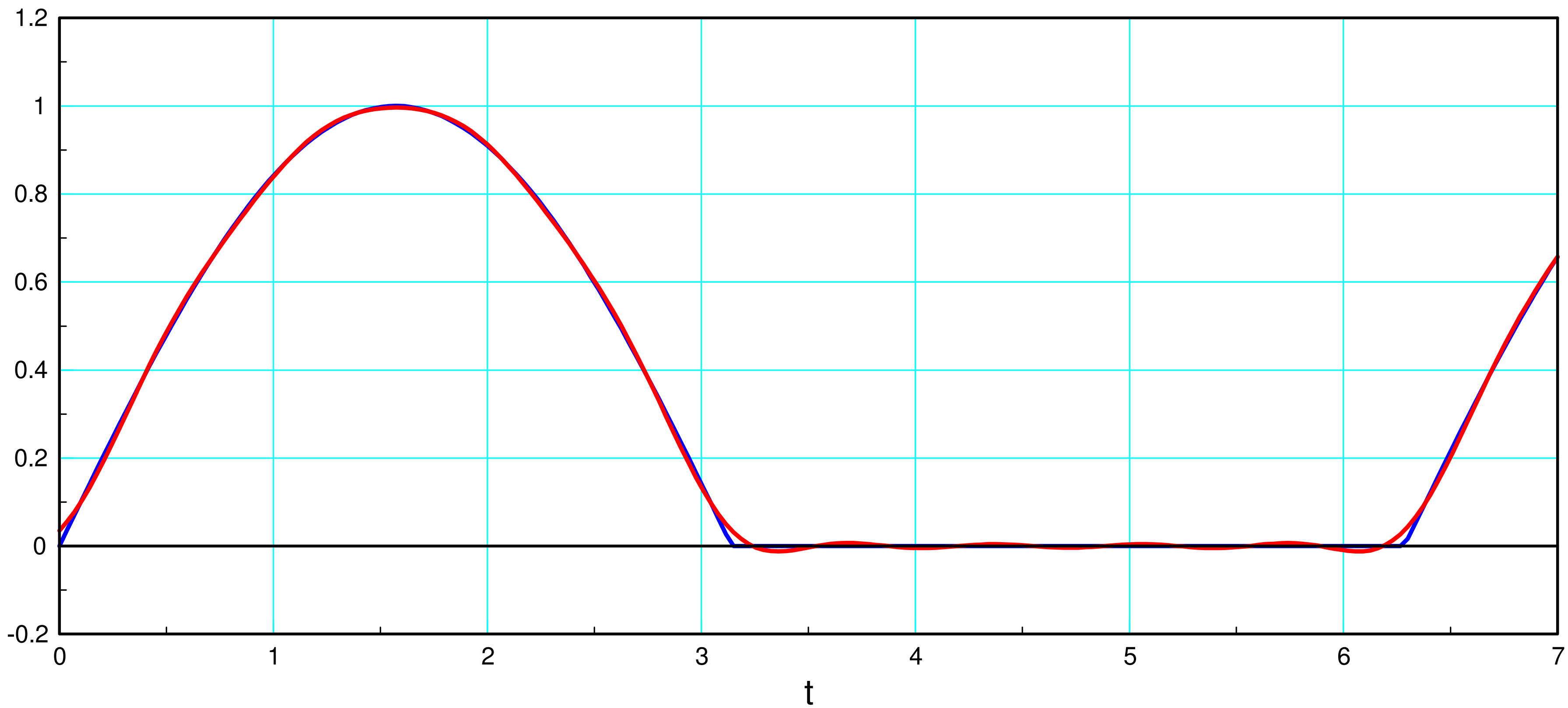
Example: Triangle Wave

- $x(t) = \frac{8}{\pi^2} \sum_{\text{n odd}}^{\infty} \frac{1}{n^2} \cos(nt)$
- n = 9 (red) & infinity (blue)



Example: 1/2 Wave Rectified Sine Wave

- $x(t) = \frac{1}{\pi} + \frac{1}{2}\sin(x) - \frac{2}{\pi}\sum_{\text{n even}} \frac{1}{n^2-1}\cos(nt)$
- n = 9 (red) & infinity (blue)



Problem: How to go from left to right

Solution #1: Least Squares

- x(t) is periodic in $T = \left(\frac{2\pi}{10}\right)$ seconds
- Let $\omega_0 = \left(\frac{2\pi}{T}\right)$ = 10 rad/sec

Approximate I(t) as

$$I(t) \approx a_0 + a_1 \cos(10t) + b_1 \sin(10t) + a_2 \cos(20t) + b_2 \sin(20t) + \ldots$$

Lease Squares Solution:

$$I(t) \approx \begin{bmatrix} 1 & \cos(10t) & \sin(10t) & \cos(20t) & \sin(20t) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ b_1 \\ a_2 \\ b_2 \end{bmatrix}$$

In matrix form

$$Y = BA$$

$$B^T Y = B^T BA$$

$$A = (B^T B)^{-1} B^T Y$$

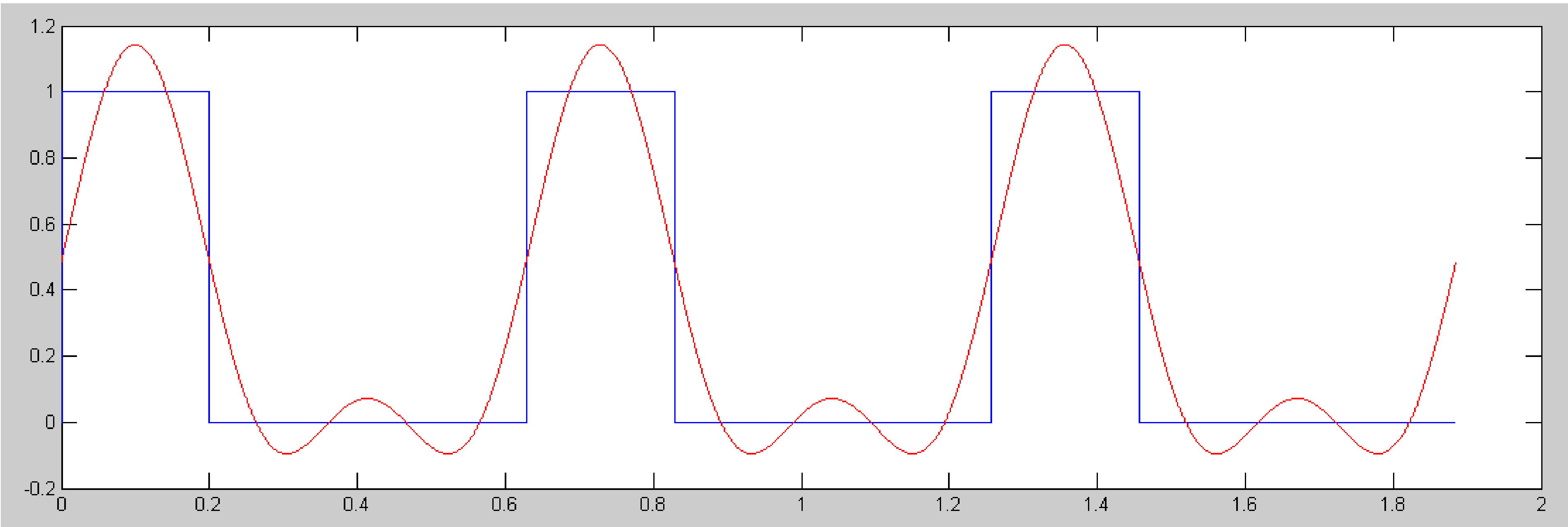
In Matlab:

```
t = [0:0.0001:1]' * 2*pi/10;
I = 1 .* (t < 0.2);

B = [t.^0, cos(10*t), sin(10*t), cos(20*t), sin(20*t)];
A = inv(B'*B)*B'*I

a0    0.3186
a1    0.2895
b1    0.4513
a2   -0.1208
b2    0.2634
```

$$I(t) \approx 0.3186 + 0.2895\cos(10t) - 0.4513\sin(10t) - 0.1208\cos(20t) - 0.2634\sin(20t)$$

Resulting Approximation

```
plot(t,I,t,B*A);
```



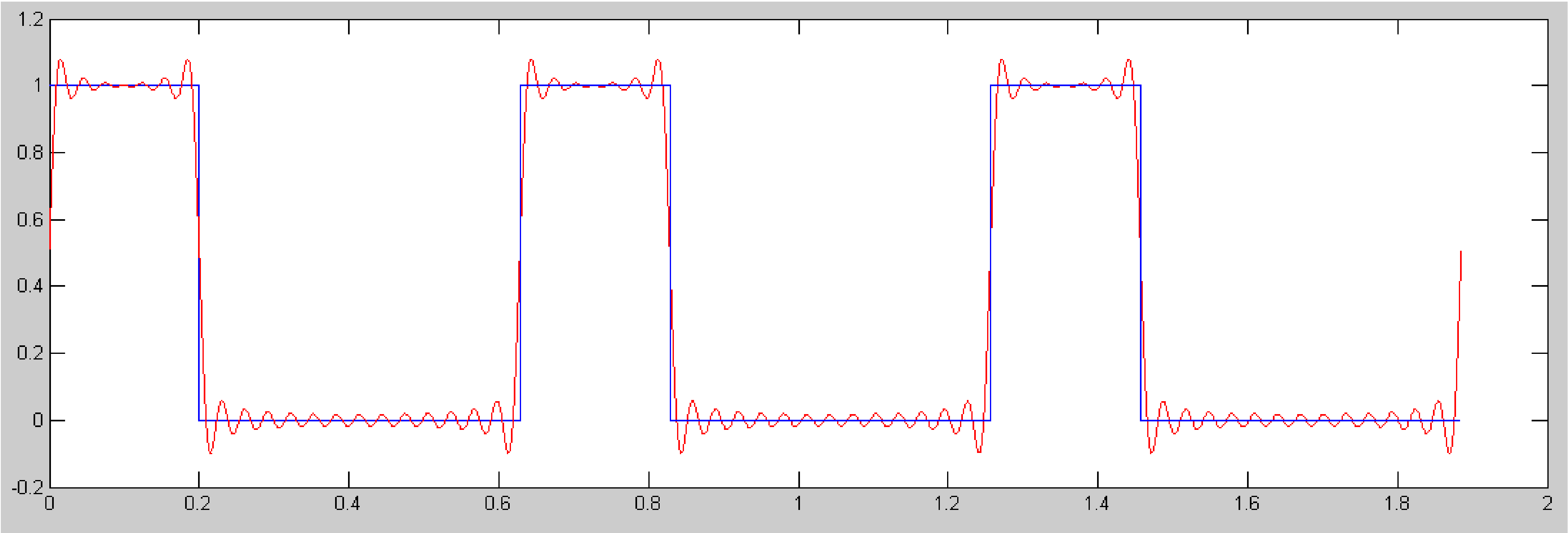
3-Cycles for I(t) (blue) and it's approximation using 5 terms (red): DC + 10 rad/sec + 20 rad/sec terms

What this means is

$$I(t) \approx 0.3186 + 0.2895\cos(10t) - 0.4513\sin(10t) - 0.1208\cos(20t) - 0.2634\sin(20t)$$

Note: The approximation gets better if you add more terms:

- 20 terms



3-Cycles for I(t) (blue) and it's approximation using 20 harmonics (40 terms + DC - red)

Solution 2: Fourier Transform

Express I(t) as

$$I(t) \approx a_0 + a_{10}\cos(10t) + b_{10}\sin(10t) + a_{20}\cos(20t) + b_{20}\sin(20t)$$

a0:

$$avg(\cos(at)) = 0$$

$$avg(\sin(at)) = 0$$

You can thus determine the DC term (a0) by

$$\boxed{a_0 = avg(I(t))}$$

a10 and b10, a20 and b20, etc:

$$avg(\sin(at)\cdot\cos(bt)) = 0$$

$$avg(\sin(at)\cdot\sin(bt)) = \begin{cases} \frac{1}{2} & a = b \\ 0 & otherwise \end{cases}$$

$$avg(\cos(at)\cdot\cos(bt)) = \begin{cases} \frac{1}{2} & a = b \\ 0 & otherwise \end{cases}$$

Thus

$$\boxed{\begin{aligned} a_{10} &= 2\cdot avg(\cos(10t)\cdot I(t)) \\ b_{10} &= 2\cdot avg(\sin(10t)\cdot I(t)) \end{aligned}}$$

$$\boxed{\begin{aligned} a_{20} &= 2\cdot avg(\cos(20t)\cdot I(t)) \\ b_{20} &= 2\cdot avg(\sin(20t)\cdot I(t)) \end{aligned}}$$

In Matlab:

- Same answer as before - just easier

```
>> a0 = mean(I)

    0.3186

>> a10 = 2*mean(cos(10*t) .* I)

    0.2896

>> b10 = 2*mean(sin(10*t) .* I)

    0.4513

>> a20 = 2*mean(cos(20*t) .* I)

   -0.1207

>> b20 = 2*mean(sin(20*t) .* I)

    0.2634
```

Net Result

- Least squares and Fourier transform are the same thing

I(t) has three terms

- A DC term:

$$I_0(t) \approx 0.3186$$

- A term at 10 rad/sec

$$I_{10}(t) \approx 0.2895\cos(10t) - 0.4513\sin(10t)$$

- A term at 20 rad/sec

$$I_{20}(t) \approx 0.1208\cos(20t) - 0.2634\sin(20t)$$

	a0	a1	b1	a2	b2
Least Squares Solution	0.3186	0.2895	0.4513	-0.1208	0.2634
Fourier Transform Solution	0.3186	0.2896	0.4513	-0.1207	0.2634

Finding V(t):

Use superposition

$$f(a+b+c) = f(a) + f(b) + f(c)$$

$$V = \left(\frac{s+10}{s^2+10s+10}\right)(I_0 + I_{10} + I_{20})$$

$$= \left(\frac{s+10}{s^2+10s+10}\right) I_0 + \left(\frac{s+10}{s^2+10s+10}\right) I_{10} + \left(\frac{s+10}{s^2+10s+10}\right) I_{20}$$

Treat this as three separate problems

- w = 0 *DC term*
- w = 10 *1st harmonic*
- w = 20 *2nd harmonic*

Then add the results togetther

DC Term

- Matlab Solution:

DC

$$I_0(t) = 0.3186$$

Using phasor analysis

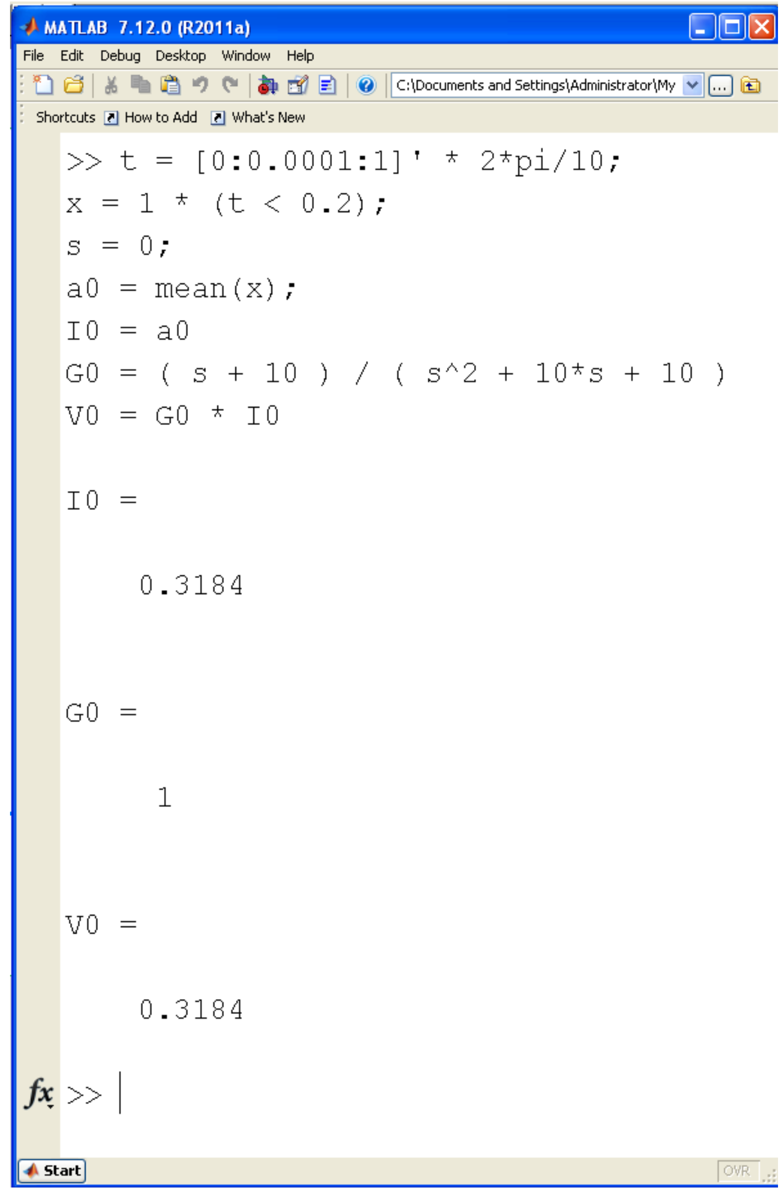
$$I = 0.3186$$

$$s = 0$$

$$V = \left(\frac{s+10}{s^2+10s+10} \right)_{s=0} \cdot I$$

$$V = (1) \cdot (0.3184)$$

$$v_0(t) = 0.3184$$



1st Harmonic

- w = 10 rad/sec
- $I_{10}(t) = 0.2895\cos(10t) - 0.4513\sin(10t)$

Using phasor analysis

$$I = 0.2895 + j0.4513$$

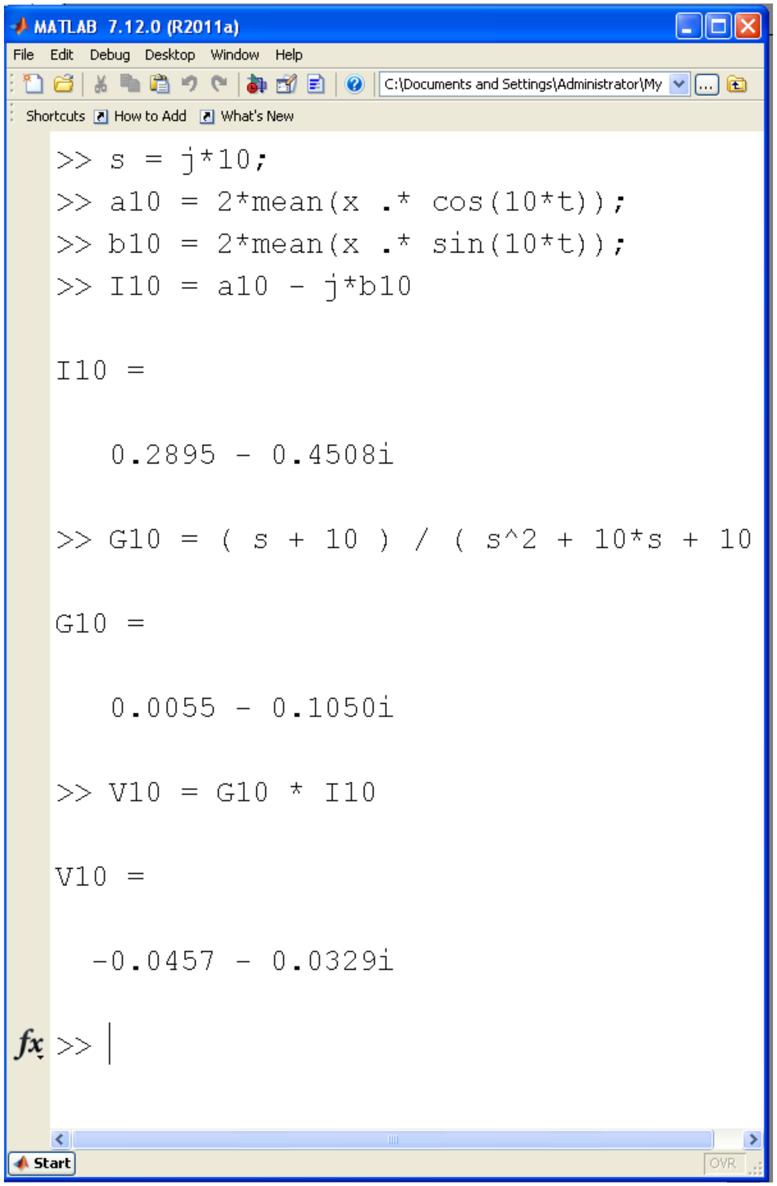
$$s = j10$$

$$V = \left(\frac{s+10}{s^2+10s+10}\right)_{s=j10} \cdot I$$

$$V = (0.0055 - j0.1050)\cdot(0.2895 + j0.4513)$$

$$V = \text{-}0.0458 \text{ - } 0.0329\text{i}$$

$$v_{10} = -0.0458\cos(10t) + 0.0329\sin(10t)$$



2nd Harmonic

- w = 20 rad/sec
- $I_{20}(t) = 0.1208\cos(20t) - 0.2634\sin(20t)$

Using phasor analysis

$$I = 0.1208 + j0.2634$$

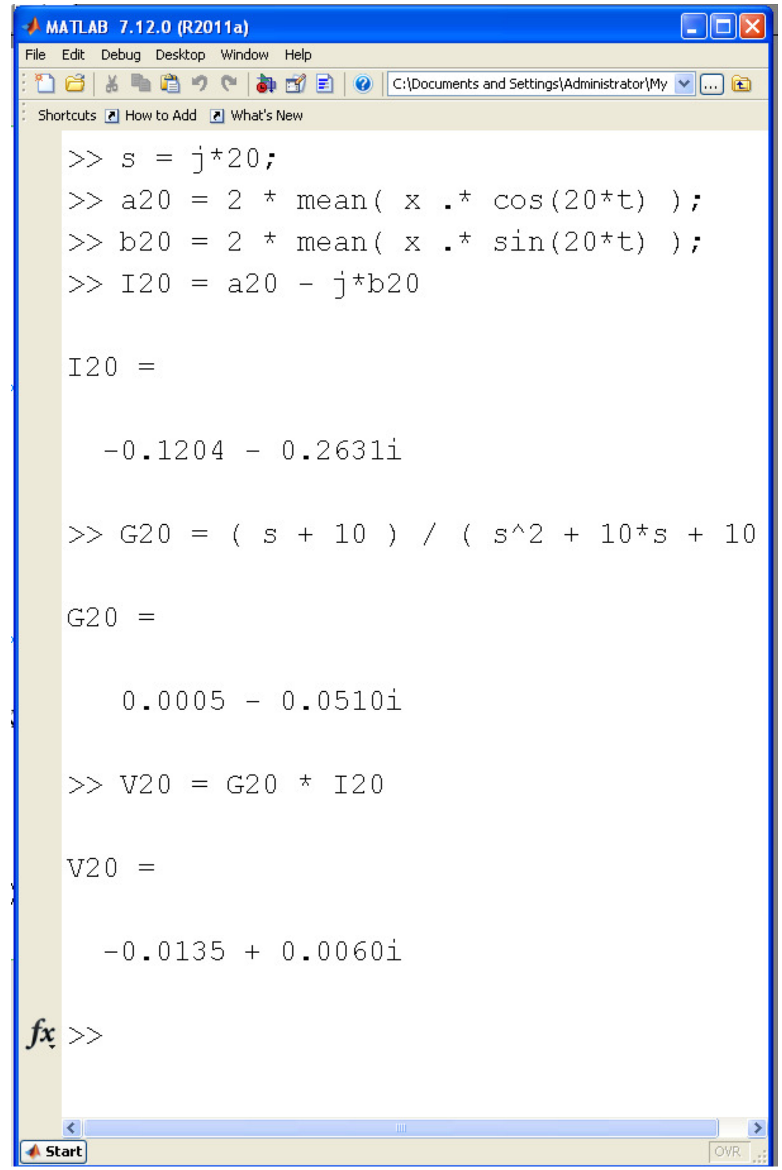
$$s = j20$$

$$V = \left(\frac{s+10}{s^2+10s+10}\right)_{s=j20} \cdot I$$

$$V = (0.0005 - j0.0510) \cdot (0.1208 + j0.2634)$$

$$V = -0.0135 + j0.0060$$

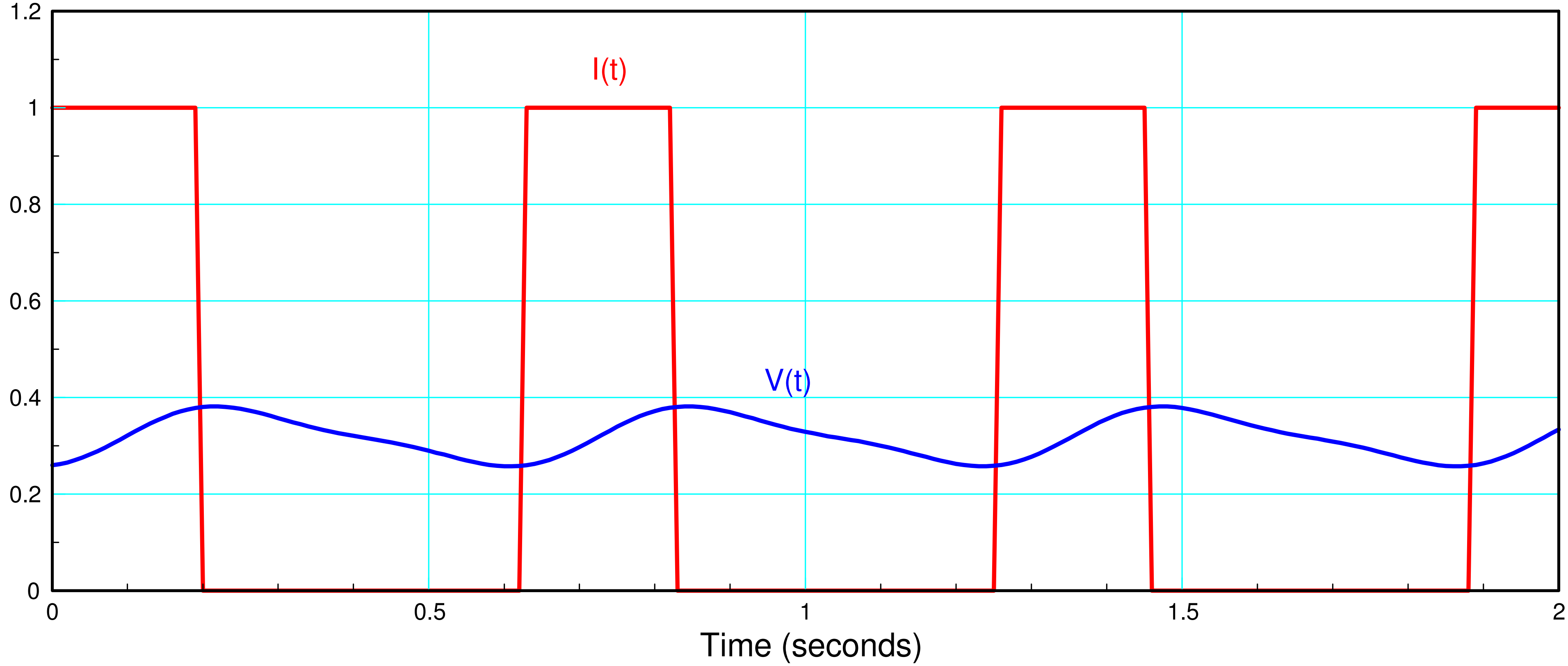
$$v_{20} = -0.0135\cos(20t) - 0.0060\sin(20t)$$



Total Answer:

$$v(t) = v_0(t) + v_{10}(t) + v_{20}(t)$$

$$v(t) = 0.3186 - 0.0458\cos(10t) + 0.0329\sin(10t) - 0.0135\cos(20t) - 0.0060\sin(20t)$$



Handout:

Find y(t)

$$Y = \left(\frac{20}{s+10}\right) X$$

$$x(t) = 2 + 3\cos(4t) + 5\sin(6t)$$

Question: How Many Terms to Include?

In practice, just a few

- Most signals, x(t), are dominated by lower-frequency terms
- Most filters, G(s), are low-pass filters
- Net result: Only a few terms are needed
- DC & 1st Harmonic are often all you need

Harmonic	% of x(t)	% of y(t)
DC	32.3980%	98.3493%
1st	45.8093%	1.5366%
2nd	13.2829%	0.1050%
3rd	0.1300%	0.0004%
4th	2.3441%	0.0045%
5th	2.3577%	0.0029%
6th	0.1277%	0.0001%

Summary:

Once you have

- Phasors, and
- Superposition

you have Fourier Transforms.

Fourier Transforms allow you to solve differential equations with *any* periodic input

- Express x(t) as the sum of a DC term and harmonics
- Find y(t) at each frequency separately
- Add the results together to get the total y(t)

Matlab makes this process fairly easy

- A little tedious if you include lots of terms
- Typically, only a few terms are sufficient