
Filters

Analysis & Design using Matlab

ECE 111 Introduction to ECE
Week #14

Please visit [Bison Academy](#) for corresponding
lecture notes, homework sets, and solutions

Filters:

A filter is any circuit whose gain varies with frequency

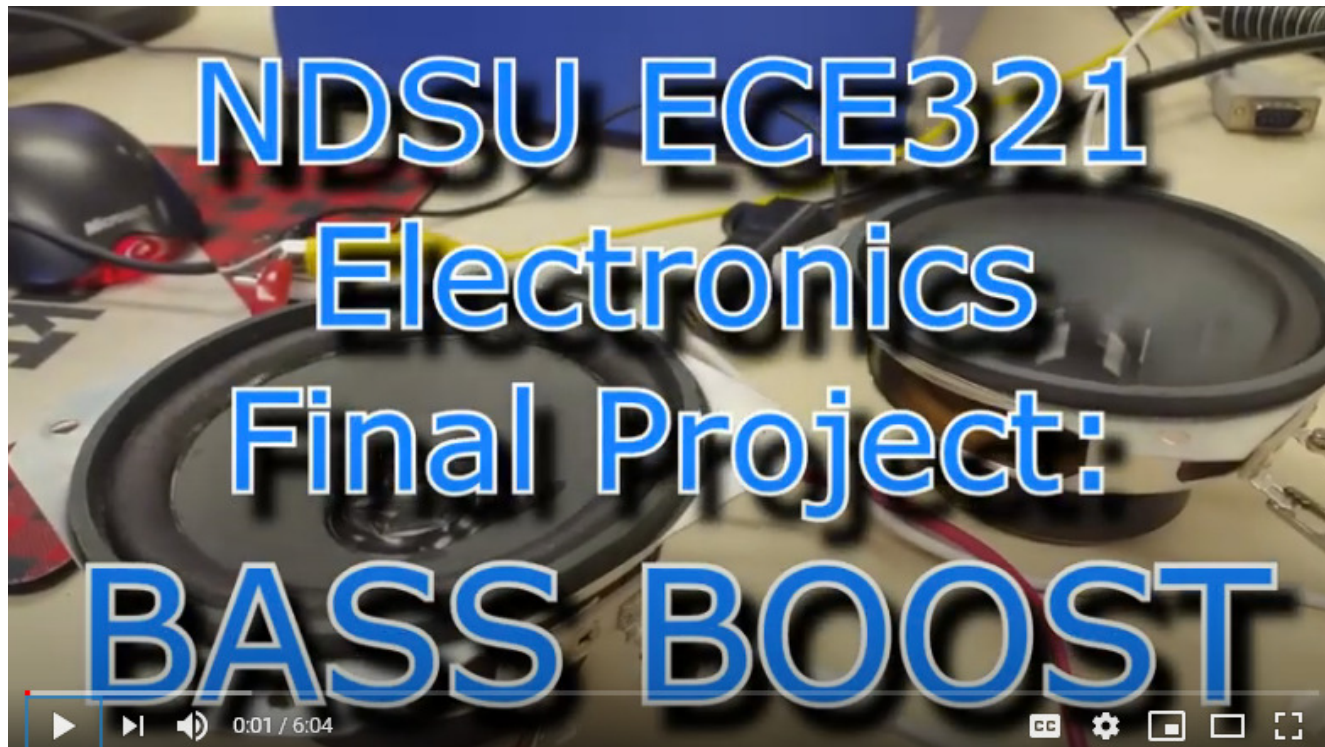
- Any circuit with inductors and/or capacitors
- Anything that satisfies a differential equation

Filter design looks at how to choose the filter to

- Pass frequencies you want, and
 - Reject frequencies you don't want.
-

Example: Bass Boost

- https://www.youtube.com/watch?v=zKfc_VoyVUM&feature=youtu.be
- Building a sub-woofer crossover
- Pass frequencies below 250Hz
- Reject frequencies above 400Hz



Differential Equations

Differential equations describe almost everything

- Why Calculus I, II, III, IV are required

Any circuit with inductors and capacitors are described by differential equations

Inductor:

$$E = \frac{1}{2}LI^2$$

$$\frac{d}{dt}(E) = P = VI = LI \frac{dI}{dt}$$

$$V = L \frac{dI}{dt}$$

Capacitor:

$$E = \frac{1}{2}CV^2 \quad \text{Joules}$$

$$\frac{d}{dt}(E) = P = VI = CV \frac{dV}{dt} \quad \text{Watts}$$

$$I = C \frac{dV}{dt}$$

Transfer Functions

Assume a 3rd-order differential equation relating x and y:

$$y''' + 4y'' + 6y' + 8y = 10x' + 30x$$

$$y' \equiv \frac{dy}{dx}$$

Assume all functions are in the form of

$$y(t) = e^{st}$$

Then

$$\frac{d}{dt}(e^{st}) = s \cdot e^{st}$$

sY means *the derivative of y(t)*

With this assumption

$$y''' + 4y'' + 6y' + 8y = 10x' + 30x$$

becomes

$$s^3 Y + 4s^2 Y + 6sY + 8Y = 10sX + 30X$$

Solving for Y:

$$Y = \left(\frac{10s+30}{s^3+4s^2+6s+8} \right) X = G(s) X$$

G(s) is called the *transfer function* of the system.

- Essentially, it is the gain from X to Y

Example: Find the differential equation relating X and Y given

$$Y = \left(\frac{10s+30}{s^3+4s^2+6s+8} \right) X$$

Solution: First, cross multiply

$$(s^3 + 4s^2 + 6s + 8)Y = (10s + 30)X$$

Next, replace each 's' with $\frac{d}{dt}$

$$y''' + 4y'' + 6y' + 8y = 10x' + 30x$$

or equivalently

$$\frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 10\frac{dx}{dt} + 30x$$

Handout

Problem 1: Determine the transfer function from the differential equation

$$y'' + 5y' + 8y = 2x' + 10x$$

Handout

Problem 2: Determine the differential equation which relates X and Y

$$Y = \left(\frac{10s+20}{s^2+6s+5} \right) X$$

Transfer Functions with DC:

Find $y(t)$:

$$Y = \left(\frac{10s+30}{s^3+4s^2+6s+8} \right) X \qquad x(t) = 2$$

Solution:

$$x(t) = 2 \cdot e^{0t} = 2$$

$$s = 0$$

$$X = 2 + j0$$

$$Y = \left(\frac{10s+30}{s^3+4s^2+6s+8} \right)_{s=0} (2 + j0) = 7.50$$

$$y(t) = 7.5$$

Transfer Function with a Sinusoidal Input

$$Y = \left(\frac{10s+30}{s^3+4s^2+6s+8} \right) X$$

$$x(t) = 2 \cos(3t)$$

Convert to phasor form

$$s = j3$$

$$X = 2 + j0$$

$$a + jb \rightarrow a \cdot \cos(\omega t) - b \cdot \sin(\omega t)$$

$$Y = \left(\frac{10s+30}{s^3+4s^2+6s+8} \right)_{s=j3} \cdot (2 + j0)$$

$$Y = -2.566 - j1.318$$

rectangular form

$$Y = 2.885 \angle -152.8^\circ$$

polar form

meaning

$$y(t) = -2.566 \cos(3t) + 1.318 \sin(3t)$$

$$y(t) = 2.885 \cos(3t - 152.8^\circ)$$

Either form is valid

Note: Answer varies with frequency

- It's a filter

Example: Find $y(t)$ for an input at 30 rad/sec:

$$Y = \left(\frac{10s+30}{s^3+4s^2+6s+8} \right) X \qquad x(t) = 2 \cos(30t)$$

Solution:

$$s = j30$$

$$X = 2 + j0$$

$$Y = \left(\frac{10s+30}{s^3+4s^2+6s+8} \right)_{s=j30} \cdot (2 + j0)$$

$$Y = (-0.0223 - j0.0007)$$

which means

$$y(t) = -0.0223 \cos(30t) + 0.0007 \sin(30t)$$

MATLAB Code:

Input the frequency for s and evaluate G(s)

```
s = 0;  
X = 2;  
Y = (10*s + 30) / (s^3 + 4*s^2 + 6*s + 8) * (2)
```

```
Y = 7.5000
```

```
s = j*3;  
X = 2 + j*0;  
Y = (10*s + 30) / (s^3 + 4*s^2 + 6*s + 8) * (2 + j*0)
```

```
Y = -2.5665 - 1.3179i
```

```
s = j*30;  
X = 2 + j*0;  
Y = (10*s + 30) / (s^3 + 4*s^2 + 6*s + 8) * (2 + j*0)
```

```
Y = -0.0223 - 0.0007i
```

You can also input $G(s)$ as a transfer function and use the MATLAB function `evalfr()`

```
G = tf([10,30],[1,4,6,8])
```

$$\frac{10s + 30}{s^3 + 4s^2 + 6s + 8}$$

```
Y = evalfr(G, 0) * 2
```

```
Y = 7.5000
```

```
Y = evalfr(G, j*3) * 2
```

```
Y = -2.5665 - 1.3179i
```

```
Y = evalfr(G, j*30) * 2
```

```
Y = -0.0223 - 0.0007i
```

which are the same answers as before.

Handout

Problem 3: Find $y(t)$

$$Y = \left(\frac{10}{(s+1)(s+3)} \right) X$$

$$x(t) = 4 \cos(5t) + 2 \sin(5t)$$

Handout

$$Y = \left(\frac{10}{(s+1)(s+3)} \right) X$$

$$x(t) = 4 \cos(5t) + 2 \sin(5t)$$

Answer:

$$s = j5$$

$$X = 4 - j2$$

$$Y = \left(\frac{10}{(s+1)(s+3)} \right)_{s=j5} (4 - j2)$$

$$Y = -1.448 - j0.407$$

meaning

$$y(t) = -1.448 \cos(5t) + 0.407 \sin(5t)$$

Frequency Response of a Filter:

- If the input is known, plug in $s = j\omega$
- For a general solution, sweep ω

Example: Determine the gain of $G(s)$ over the range of 0 to 10 rad/sec for

$$G(s) = \left(\frac{10s+30}{s^3+4s^2+6s+8} \right)$$

Option 1: Compute the gain at a bunch of points from 0 to 10 rad/sec, or

Option 2: Use MATLAB. Input the frequencies you want to evaluate:

```
w = [0:5:20]';  
s = j*w;  
G = (10*s + 30) ./ (s.^3 + 4*s.^2 + 6*s + 8);
```

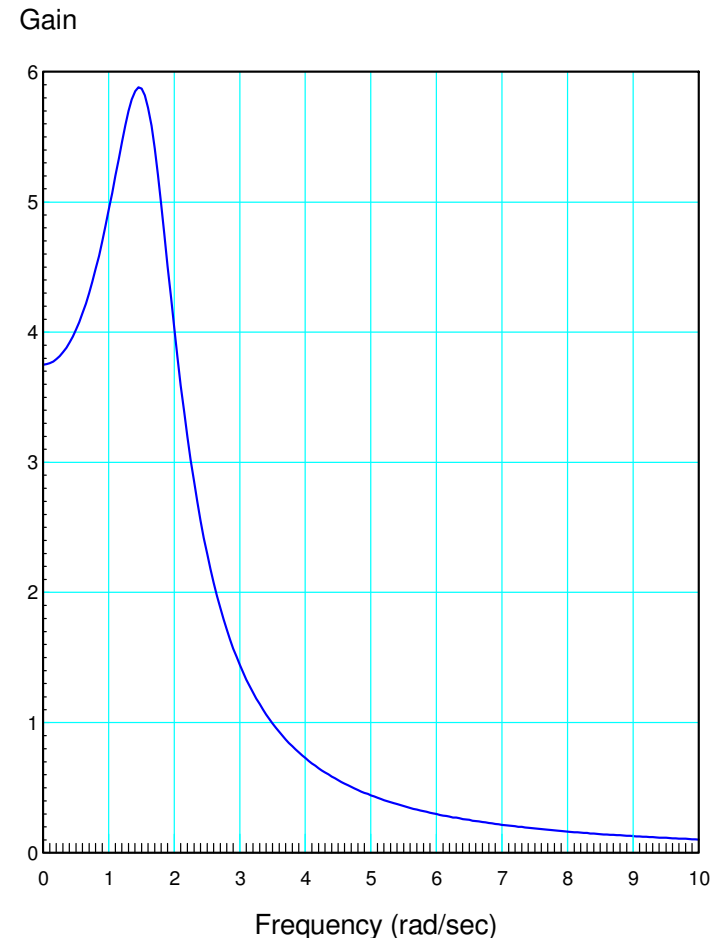
Note that dot-notation is required.

In matlab:

```
w = [0:0.05:10]';  
s = j*w;  
G = (10*s + 30) ./ (s.^3 + 4*s.^2 + 6*s +  
8);  
plot(w,abs(G));  
xlabel('Frequency (rad/sec)');  
ylabel('Gain');
```

What this graph tells you is:

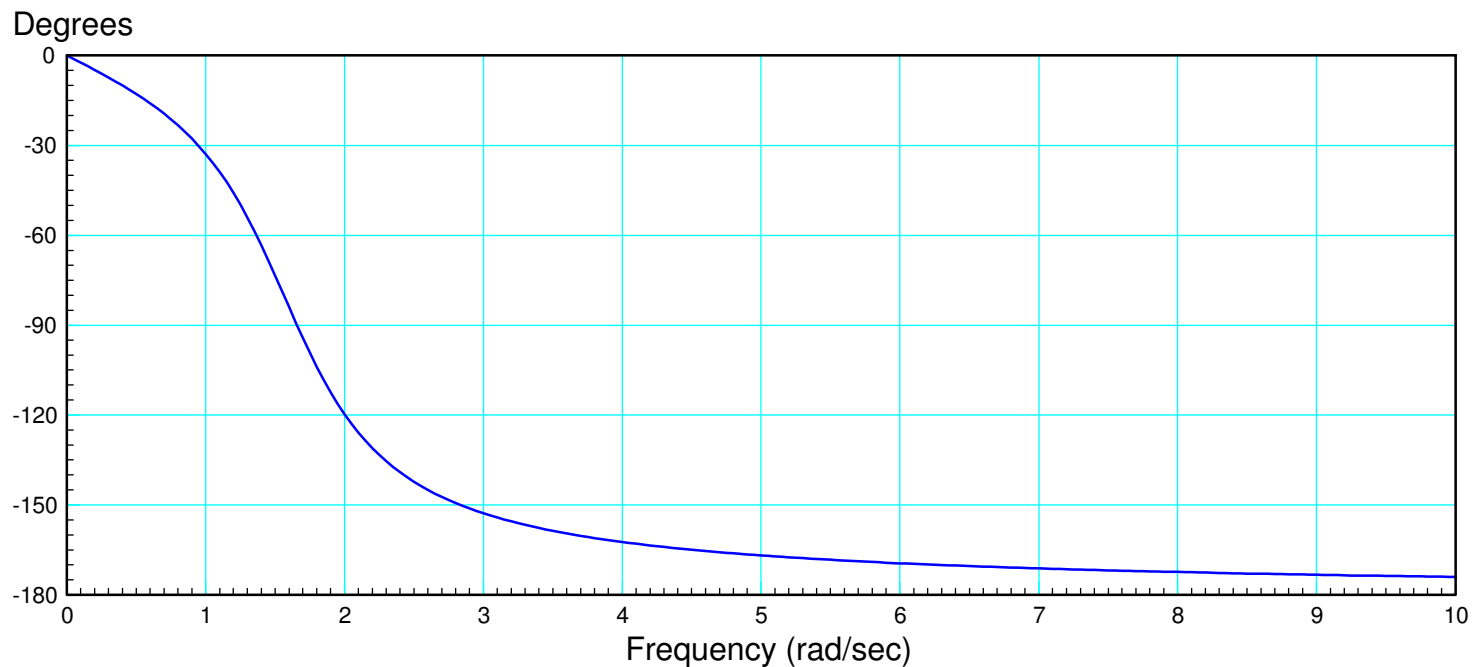
- The gain is large for frequencies below about 2 rad/sec and small for frequencies above 6 rad/sec. Since this passes low frequencies, it is called a *low-pass filter*
- The system has a resonance (a large gain) for frequencies near 1.5 rad/sec.



Phase Plot

- Not sure what this really tells you
- Usually we only deal with amplitude (gain)

```
>> plot(w, angle(G) * 180/pi);  
>> xlabel('Frequency (rad/sec)');  
>> ylabel('Angle (degrees)');
```

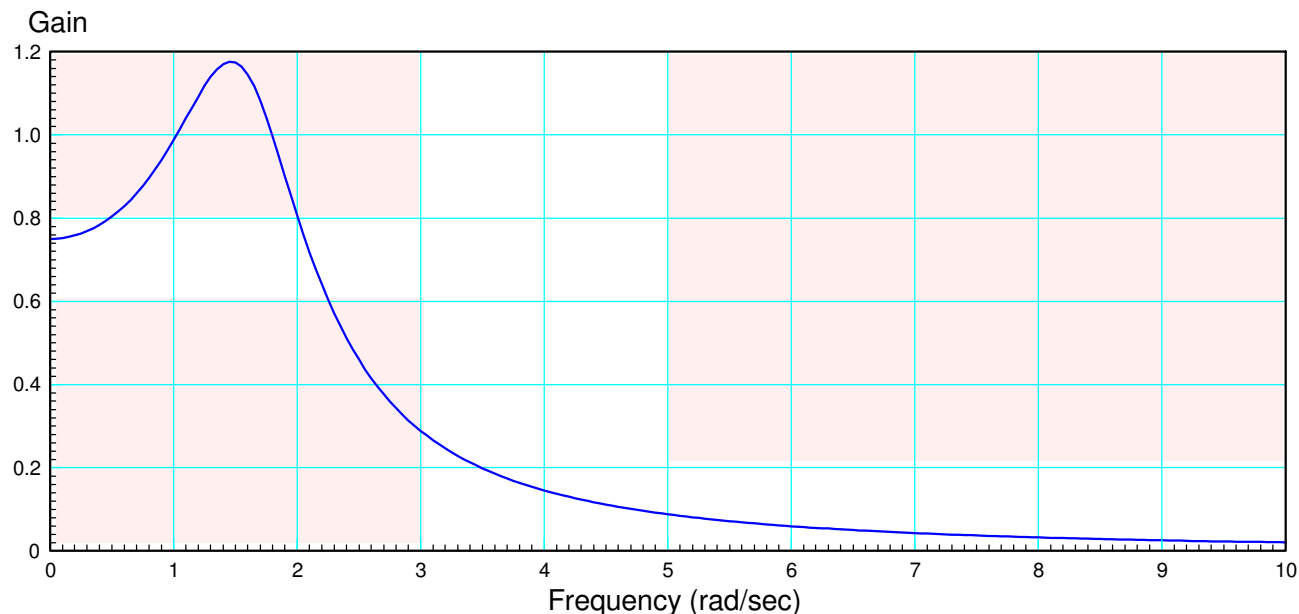


fminsearch() and m-files

Problem: How to design a filter?

- What is a 'good' transfer function?
- Covered in ECE 343 & ECE 321

Knowing nothing about filter design, you can still design a filter using Matlab



fminsearch()

- Really useful Matlab function
- Finds the minimum of a function

Example: Find $\sqrt{2}$

```
function [ J ] = cost( z )  
    e = z*z - 2;  
    J = e^2;  
end
```

Minimize in Matlab

```
>> [a,b] = fminsearch('cost',4)
```

```
a =      1.4143  
b =      1.5665e-008
```

Example: Shape of a hanging chain

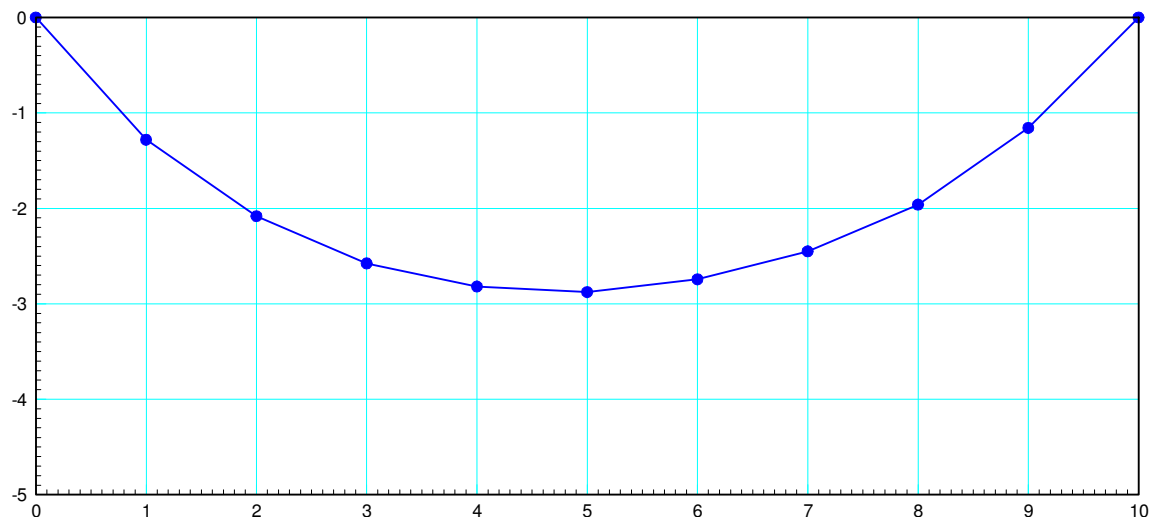
Minimize the potential energy

$$PE = mg(y_1 + y_2 + \dots + y_9)$$

Constrain the length to be 12 meters (ish)

$$J = PE + \alpha(12 - L)^2$$

```
function [ J ] = cost_chain( Z )  
  
    Y = [0;Z;0];  
    PE = sum(Y);  
    L = 0;  
    for i=2:11  
        L = L + sqrt(1 + (Y(i) - Y(i-1))^2);  
    end  
  
    E = (12 - L);  
    J = PE + 100*E^2;  
  
    plot([0:10],Y,'.-');  
    ylim([-5,1]);  
    pause(0.01);  
  
end  
  
y = i .* (i-10) / 5;  
[a,b] = fminsearch('cost',y)
```



Filter Design with fminsearch:

$$|G_d(s)| = \begin{cases} 1 & \omega < 3 \\ 0 & \omega > 3 \end{cases}$$

Step 1: Assume the form of the filter

$$G(s) = \left(\frac{a}{(s^2 + bs + c)(s^2 + ds + e)} \right)$$

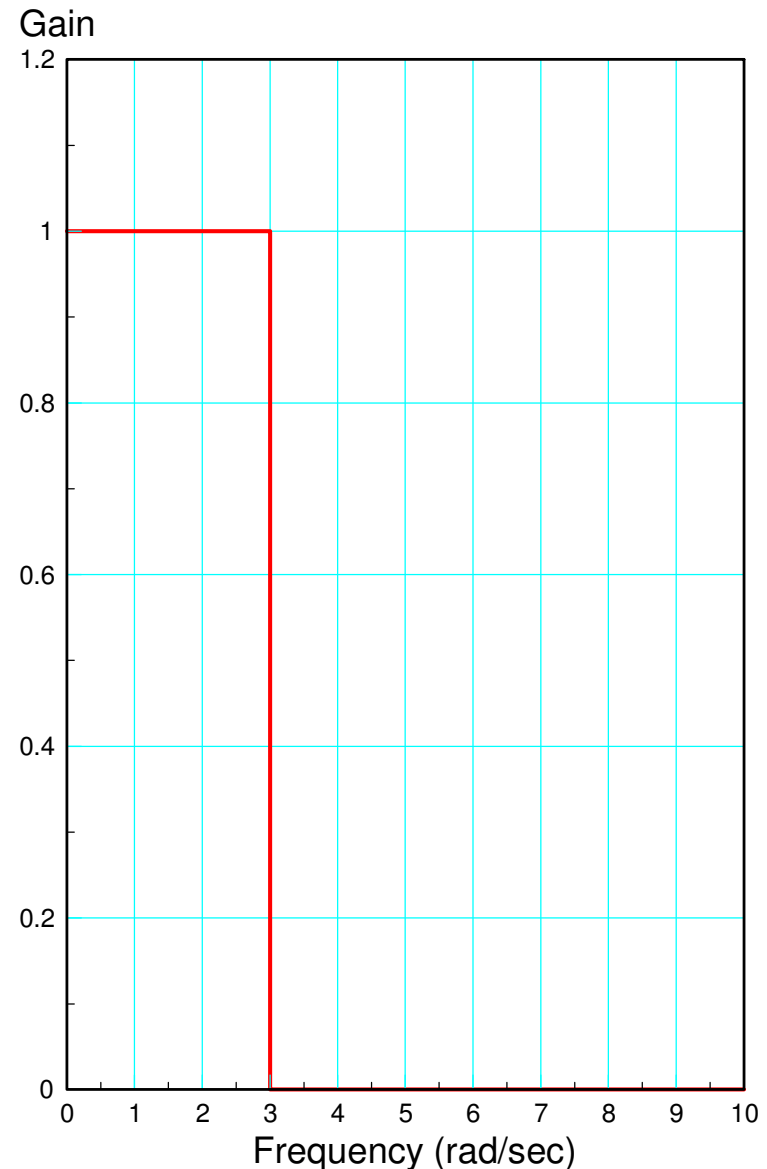
Define the cost (J)

- Minimum is when $G(s) = \text{desired filter}$

$$E(s) = |G(s)| - |G_d(s)|$$

$$J = \sum E^2$$

Guess {a, b, c, d, e} to minimize J



```
function [ J ] = costF( z )

    a = z(1);
    b = z(2);
    c = z(3);
    d = z(4);
    e = z(5);

    w = [0:0.1:10]';
    s = j*w;

    Gideal = 1 * (w < 3);

    G = a ./ ( (s.^2 + b*s + c) .* (s.^2 + d*s + e) );

    e = abs(Gideal) - abs(G);

    J = sum(e.^2);

    plot(w,abs(Gideal),w,abs(G));
    ylim([0,1.2]);
    pause(0.01);

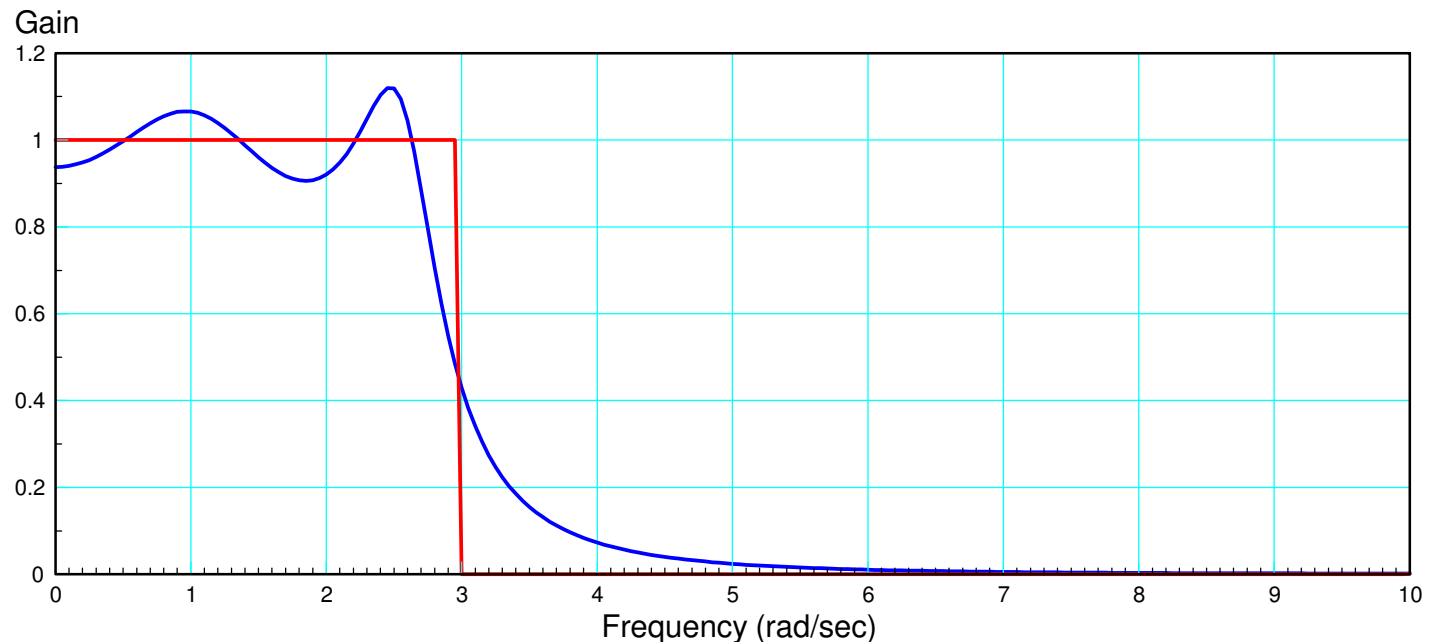
end
```

Call fminsearch with an initial guess for (a,b,c,d)

```
>> [Z,e] = fminsearch('costF',[1,2,3,4,5])
```

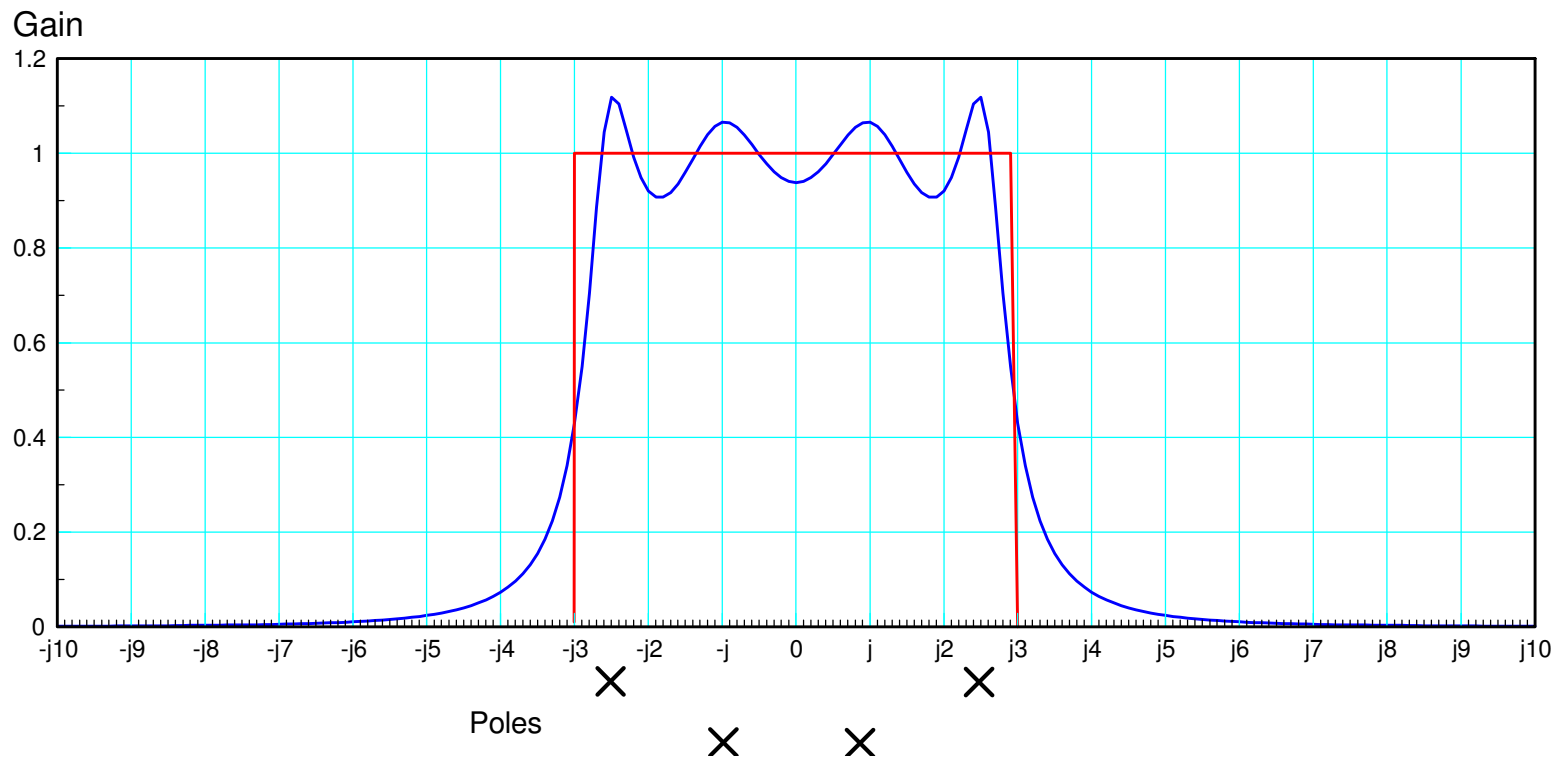
	a	b	c	d	e
Z =	10.9474	1.6224	1.7317	0.6141	6.7413
e =	0.9575				

$$G(s) = \left(\frac{10.9474}{(s^2 + 1.6224s + 1.7317)(s^2 + 0.6141s + 6.7413)} \right)$$



Sidelight: The filter isn't arbitrary.

- When you're close to a zero, the gain is small (multiply by a small number)
- When you're close to a pole, the gain is large (divide by a small number)
- Poles are $\{ s = -0.8112 \pm j1.0362, s = -0.3071 \pm j2.5782 \}$



Example 2: Design a filter to match

Assume

$$G(s) = \left(\frac{a}{(s+b)(s^2+cs+d)(s^2+ef+g)} \right)$$

Use a piecewise linear model for G_{ideal}

```
w = [0:0.1:10]';  
s = j*w;  
  
Gideal = (0.2667*w+0.2) .* (w < 3)  
+ (1.6 - 0.2*w) .* (w >= 3) .* (w < 6);
```



Matlab Function:

```
function [ J ] = costF( z )
```

```
    a = z(1);
```

```
    b = z(2);
```

```
    c = z(3);
```

```
    d = z(4);
```

```
    e = z(5);
```

```
    f = z(6);
```

```
    w = [0:0.1:10]';
```

```
    s = j*w;
```

```
    Gideal = (0.2667*w+0.2) .* (w<3) + (1.6 - 0.2*w) .* (w >= 3) .* (w<6);
```

```
    G = a ./ ( (s+b) .* (s.^2 + c*s + d) .* (s.^2 + e*s + f) );
```

```
    e = abs(Gideal) - abs(G);
```

```
    J = sum(e.^2);
```

```
    plot(w,abs(Gideal),w,abs(G));
```

```
    ylim([0,1.2]);
```

```
    pause(0.01);
```

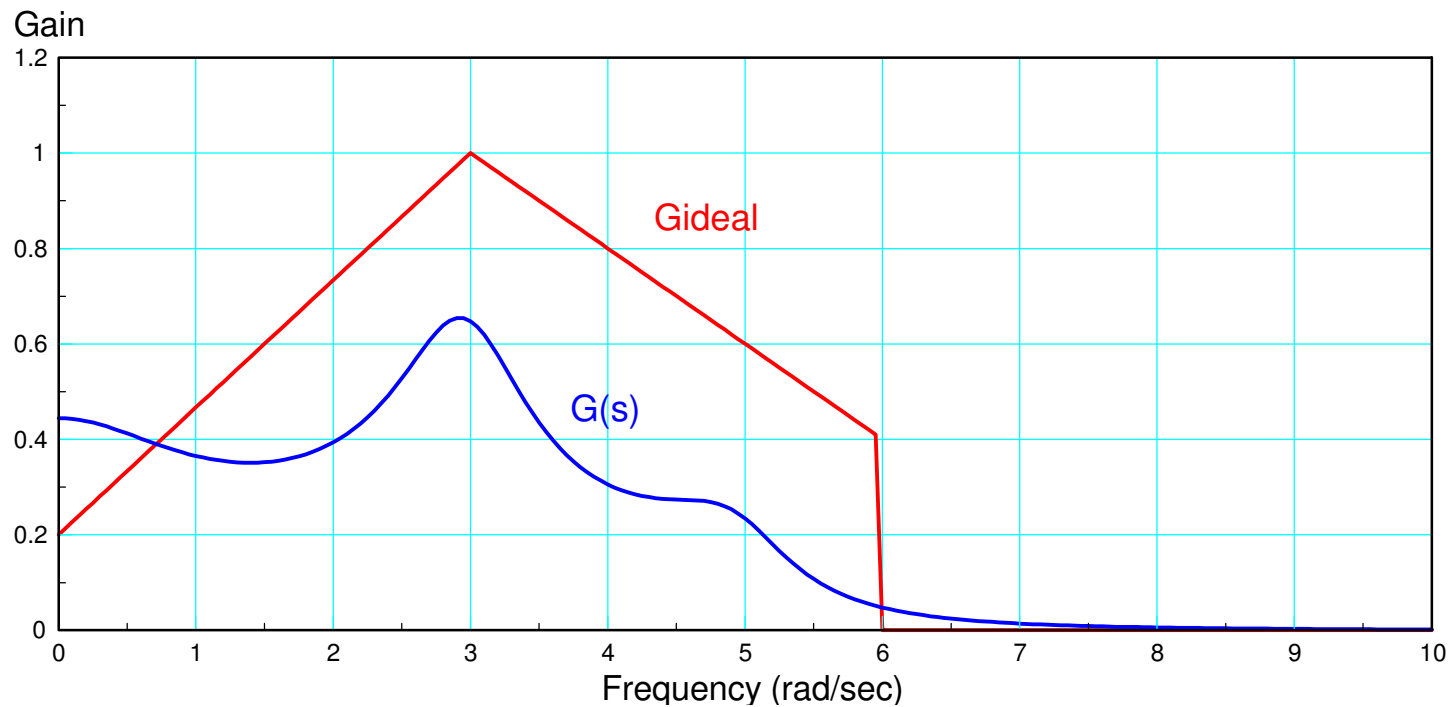
```
end
```

Optimization by hand

```
>> costF([1,2,3,4,5,6])  
ans =    27.6268
```

```
>> costF([100,1,2,9,2,25])  
ans =    13.1412
```

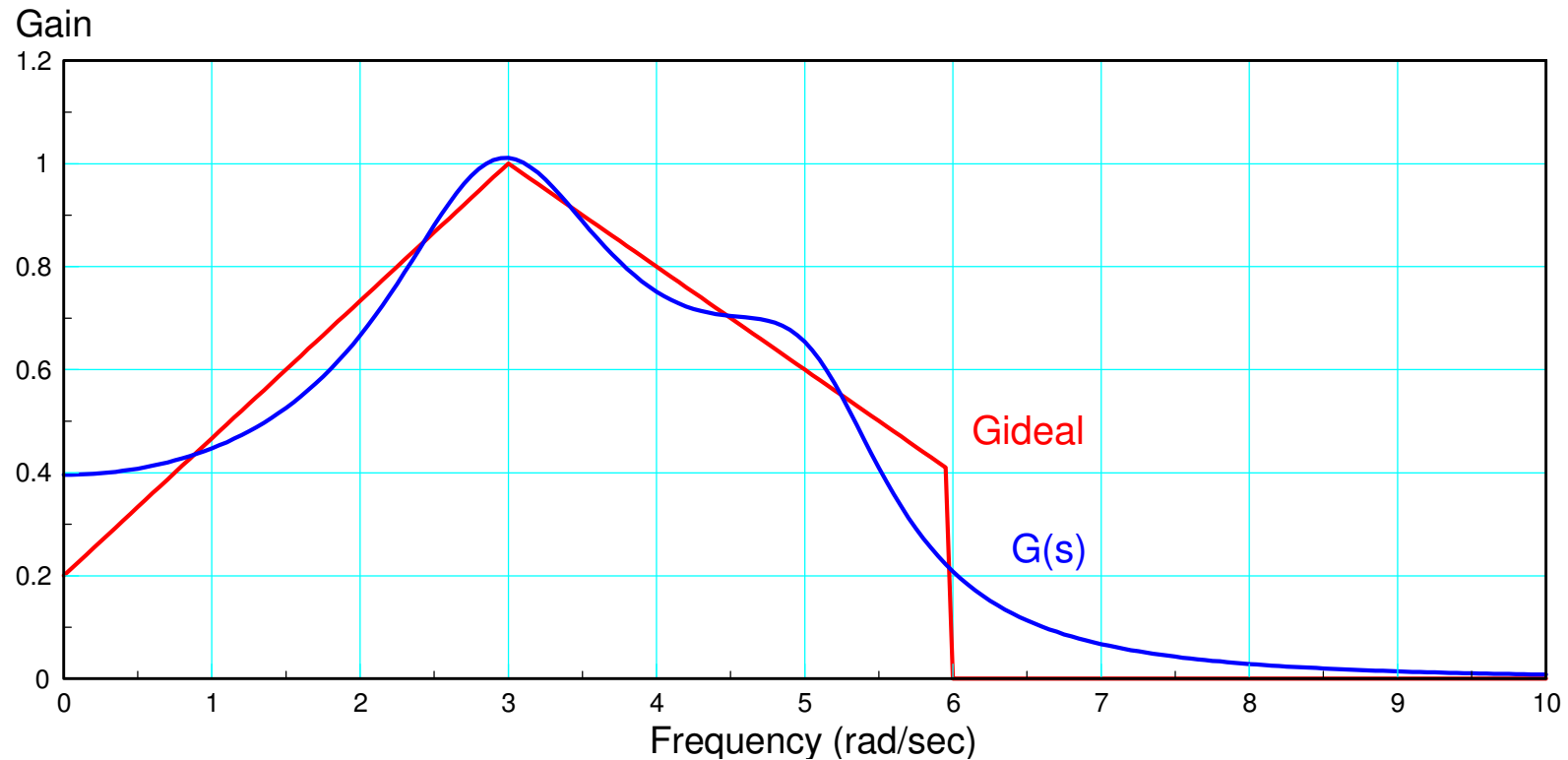
```
>> costF([100,1,1,9,1,25])  
ans =     7.1906
```



Optimization with *fminsearch*()

```
>> [Z,e] = fminsearch('costF',[100,1,1,9,1,25])
```

	a	b	c	d	e	f
Z =	651.9876	6.7179	1.6175	9.2075	1.3025	26.6229
e =	0.5270					



Summary:

A filter is a circuit where the gain depends upon frequency

- Any circuit with inductors and/or capacitors

Filter analysis is easy with complex numbers

- Plug in $s \rightarrow j\omega$
- Use phasors to represent the input and output

Filter design is harder

- Matlab's *fminsearch()* allows you to design pretty good filters even if you know nothing about filter design
 - Other methods exist and are covered in Analog Electronics and Signals & Systems
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