

Fourier Transforms

Solving differential equations when the input is periodic

Topics

- Determine a least-squares curve fit for a function using a sinusoidal series (Fourier series)
- Determine the output of a filter when the input is periodic (but not a sine wave)

Problem:

One model for the cardiovascular system is a Winkessel model:

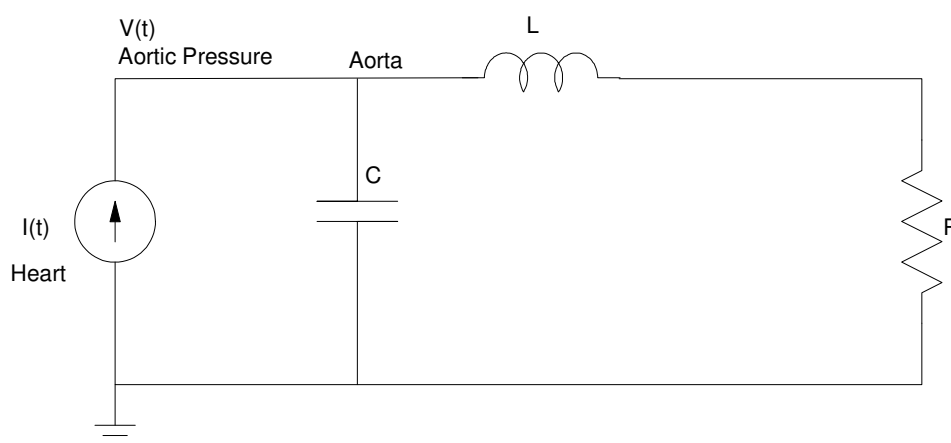


Figure 1: RLC circuit to model the cardiovascular system (Winkessel model)

Assume you know $I(t)$ (corresponding to the blood the heart pumps out). Find $V(t)$.

Using s -notation, the impedance of R , L , and C become

$$R \rightarrow R$$

$$L \rightarrow j\omega L = Ls$$

$$C \rightarrow \frac{1}{j\omega C} = \frac{1}{Cs}$$

The resistance of the above circuit is then

$$Z = \left(\frac{1}{1/Cs} + \frac{1}{Ls+R} \right)^{-1}$$

$$Z = \left(Cs + \frac{1}{Ls+R} \right)^{-1}$$

$$Z = \left(\frac{Cs(Ls+R)+1}{Ls+R} \right)^{-1}$$

$$Z = \left(\frac{Ls+R}{CLs^2+CRs+1} \right)$$

Meaning

$$V = IR$$

$$V = \left(\frac{Ls+R}{CLs^2+CRs+1} \right) I$$

If you assume

- $R = 1$
- $L = 0.1$
- $C = 1$

you get

$$V = \left(\frac{s+10}{s^2+10s+10} \right) I$$

Case 1: Sinusoidal Input

Find $V(t)$ assuming

$$I(t) = 3 \sin(10t)$$

This is a phasor problem. Using phasor notation, voltages and currents are represented as

$$a + jb \rightarrow a \cos(\omega t) - b \sin(\omega t)$$

so

$$I(t) = 0 - j3$$

The frequency is 10 rad/sec

$$s = j10$$

so the output is then

$$V = \left(\frac{s+10}{s^2+10s+10} \right) I$$

$$V = \left(\frac{s+10}{s^2+10s+10} \right)_{s=j10} \cdot (-j3)$$

$$V = (0.0055 - j0.1050) \cdot (-j3)$$

$$V = -0.3149 - 0.0166i$$

meaning

$$v(t) = -0.3149 \cos(10t) + 0.0166 \sin(10t)$$

or if you prefer polar form

$$V = 0.3154 \angle -177^\circ$$

meaning

$$v(t) = 0.3154 \cos(10t - 177^\circ)$$

Either answer is valid.

Case 2: Input is Periodic but Not a Sinusoid

Suppose instead $I(t)$ is periodic every $\frac{2\pi}{10}$ seconds

$$I\left(t + \frac{2\pi}{10}\right) = I(t)$$

and

$$I(t) = \begin{cases} 1 & 0 < t < 0.2 \\ 0 & \text{otherwise} \end{cases}$$

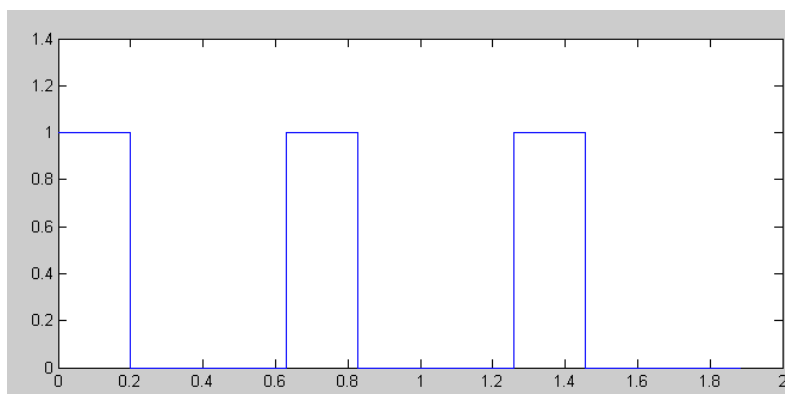


Figure 2: Approximate $I(t)$ with a square wave. Note that $I(t)$ is period every $\frac{2\pi}{10}$ seconds

Find $V(t)$ assuming $I(t)$ and $V(t)$ are related by

$$V = \left(\frac{s+10}{s^2+10s+10} \right) I$$

Now find $V(t)$

Solution 1: Least Squares Curve Fitting:

To solve this problem, change it so that $I(t)$ is a sinusoid, or more accurately, a bunch of sinusoids. Then you can use the previous solution for each sinusoidal input. Since $I(t)$ is periodic in $\frac{2\pi}{10}$ seconds, express $I(t)$ in terms of sine waves which are also periodic in $\frac{2\pi}{10}$ seconds (i.e. harmonics)

$$I(t) \approx a_0 + a_1 \cos(10t) + b_1 \sin(10t) + a_2 \cos(20t) + b_2 \sin(20t) + \dots$$

In theory, you should go out to infinity. For the sake of space, let's just go out to 5 terms.

Using least squares, we can solve for the constants.

First, write this in matrix form

$$I(t) \approx \begin{pmatrix} 1 & \cos(10t) & \sin(10t) & \cos(20t) & \sin(20t) \end{pmatrix} \begin{bmatrix} a_0 \\ a_1 \\ b_1 \\ a_2 \\ b_2 \end{bmatrix}$$

or in matrix form

$$Y = BA$$

where B is your basis function and A is your unknown constants. B is not invertable, so multiply both sides by B^T

$$B^T Y = B^T B A$$

Solve for A

$$A = (B^T B)^{-1} B^T Y$$

In Matlab:

```
>> t = [0:0.0001:1]' * 2*pi/10;
>> I = 1 .* (t < 0.2);

>> B = [t.^0, cos(10*t), sin(10*t), cos(20*t), sin(20*t)];
>> A = inv(B'*B) * B'*I

a0      0.3186
a1      0.2895
b1      0.4513
a2     -0.1208
b2      0.2634

>> plot(t, I, t, B*A);
```

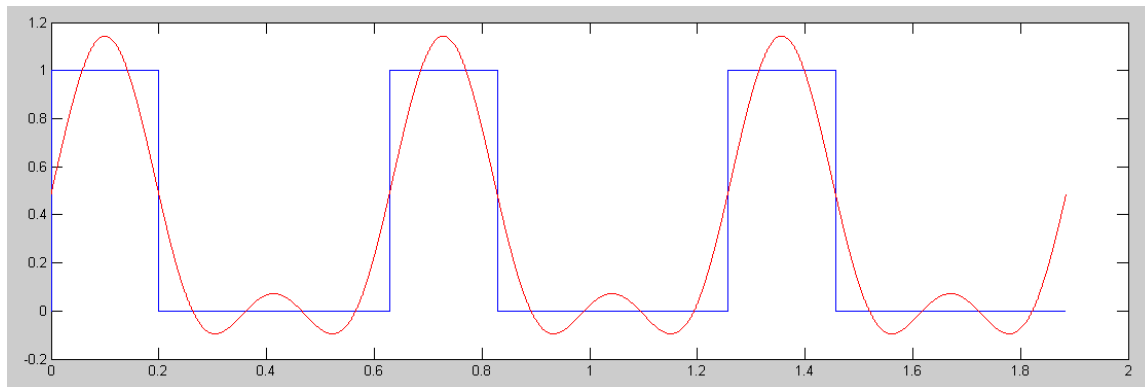


Figure 3: 3-Cycles for $I(t)$ (blue) and its approximation using 5 terms (red): DC + 10 rad/sec + 20 rad/sec terms

What this means is

$$I(t) \approx 0.3186 + 0.2895 \cos(10t) - 0.4513 \sin(10t) - 0.1208 \cos(20t) - 0.2634 \sin(20t)$$

Note: The approximation gets better if you add more terms. If you go out to the 20th harmonic, this plot looks like the following:

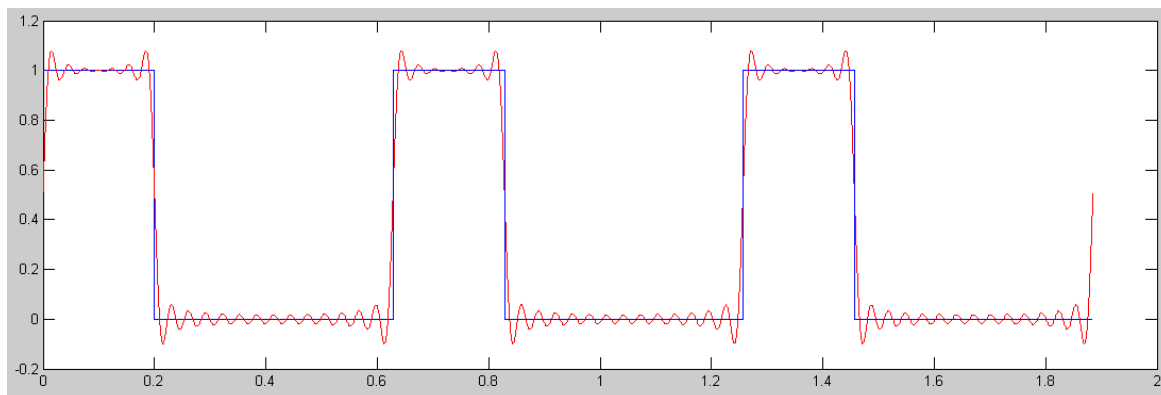


Figure 4: 3-Cycles for $I(t)$ (blue) and its approximation using 20 harmonics (40 terms + DC - red)

Adding more terms makes the analysis more accurate - but more tedious. It's kind of a judgement call: how many terms you need to include for the results to be accurate enough.

Often times, just a few harmonics are used:

- Signals tend to have most of their energy in the low-frequency terms (DC and 1st harmonic typically)
- Most systems are low-pass filters: they tend to attenuate high-frequency signals.

This results in the higher harmonics starting out small and getting smaller after filtering. It looks like you need a lot of terms to do Fourier analysis - but in actuality you normally only need a few (1 to 3).

Solution 2: Fourier Transform

There's actually an easier way to get each term. Assume that

$$I(t) \approx a_0 + a_{10}\cos(10t) + b_{10}\sin(10t) + a_{20}\cos(20t) + b_{20}\sin(20t)$$

Note that

$$\text{avg}(\cos(at)) = 0$$

$$\text{avg}(\sin(at)) = 0$$

You can thus determine the DC term (a_0) by

$$a_0 = \text{avg}(I(t))$$

Also note that all sine waves are orthogonal

$$\text{avg}(\sin(at) \cdot \cos(bt)) = 0$$

$$\text{avg}(\sin(at) \cdot \sin(bt)) = \begin{cases} \frac{1}{2} & a = b \\ 0 & \text{otherwise} \end{cases}$$

$$\text{avg}(\cos(at) \cdot \cos(bt)) = \begin{cases} \frac{1}{2} & a = b \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$\begin{aligned} a_{10} &= 2 \cdot \text{avg}(\cos(10t) \cdot I(t)) \\ b_{10} &= 2 \cdot \text{avg}(\sin(10t) \cdot I(t)) \end{aligned}$$

$$\begin{aligned} a_{20} &= 2 \cdot \text{avg}(\cos(20t) \cdot I(t)) \\ b_{20} &= 2 \cdot \text{avg}(\sin(20t) \cdot I(t)) \end{aligned}$$

etc. In Matlab:

```
>> a0 = mean(I)

0.3186

>> a10 = 2*mean(cos(10*t) .* I)

0.2896

>> b10 = 2*mean(sin(10*t) .* I)

0.4513

>> a20 = 2*mean(cos(20*t) .* I)
```

```

-0.1207

>> b20 = 2*mean(sin(20*t) .* I)

0.2634

```

Note that you get the exact same answer you got using least squares curve fitting. So, $I(t)$ contains three frequencies (five terms):

- A DC term:

$$I_0(t) \approx 0.3186$$

- A term at 10 rad/sec

$$I_{10}(t) \approx 0.2895 \cos(10t) - 0.4513 \sin(10t)$$

- A term at 20 rad/sec

$$I_{20}(t) \approx 0.1208 \cos(20t) - 0.2634 \sin(20t)$$

In Matlab:

```

% Start with the DC term...
I0 = a0;

% add in the 10 rad/sec term...
I10 = a10*cos(10*t) + b10*sin(10*t);

% add in the 20 rad/sec term...
I20 = a20*cos(20*t) + b20*sin(20*t);

% plot I(t) alongs with it's approximation taken out to 20 rad/sec
plot(t,I,t,I0 + I10 + I20)

```

This gives same results as before.

	a0	a1	b1	a2	b2
Least Squares Solution	0.3186	0.2895	0.4513	-0.1208	0.2634
Fourier Transform Solution	0.3186	0.2896	0.4513	-0.1207	0.2634

Table 1: The weightings for each sine wave can be computed using least-squares or a formula. The results are the same.

Fourier Transforms is just curve fitting where you use sinusoids for the basis.

Finding V(t):

Using Fourier Transforms, you can convert a periodic signal into a sum of sinusoids. For our system

$$V = \left(\frac{s+10}{s^2+10s+10} \right) I$$

By using Fourier Transforms, I(t) can be expressed as

$$I(t) \approx 0.3186 + 0.2895 \cos(10t) - 0.4513 \sin(10t) - 0.1208 \cos(20t) - 0.2634 \sin(20t)$$

To find V(t), treat this as three separate problems: one at each frequency.

- $I_1(t) = 0.3186$
- $I_2(t) = 0.2895 \cos(10t) - 0.4513 \sin(10t)$
- $I_3(t) = -0.1208 \cos(20t) - 0.2634 \sin(20t)$

Phasor analysis allows us to find V(t) for each separate input. Add up the results and you have the total output, V(t).

To do this, set up a table: one column for each frequency. Analyze the circuit for each input separately

	I1(t)	I2(t)	I3(t)
Frequency: s	s = 0	s = j10	s = j20
i(t)	0.3186	0.2895 cos(10t) - 0.4513 sin(10t)	-0.1208 cos(20t) - 0.2634 sin(20t)
I(s): Phasor representation for i(t) note: a + jb means a cos(wt) - b sin(wt)	0.3186	0.2895 + j0.4513	-0.1201 + j0.2634
$G(s) = \left(\frac{s+10}{s^2+10s+10} \right)_{s=j\omega}$	1.000	0.0055 - j0.1050	0.0005 - j0.0510
V(s) = G(s) * I(s) phasor representation for v(t)	0.3186	0.0490 - j0.0279	0.0135 - j0.0060
v(t)	0.3186	0.0490 cos(10t) + 0.0279 sin(10t)	0.0135 cos(20t) - 0.0060 sin(20t)

Table 2: When I(t) contains three different frequencies, treat this as three separate problems: one at each frequency

If you add up all of the inputs (I1(t) + I2(t) + I3(t)), you get the total input.

If you add up all the outputs, you get the total output

$$v(t) = 0.3186 + 0.0490 \cos(10t) + 0.0279 \sin(10t) + 0.0135 \cos(20t) - 0.0060 \sin(20t)$$

Once you understand phasors, you can analyze any circuit or differential equation with sinusoidal inputs. Fourier Transforms are a way to convert any periodic signal into a sum of sinusoidal inputs.

A step-by-step procedure to analyze this circuit at each frequency follows. It's really just filling in the above table...

Detailed Analysis at Each Frequency

Case 1:

$$I(t) = 0.3186$$

Using phasor analysis

$$I = 0.3186$$

$$s = 0$$

$$V = \left(\frac{s+10}{s^2+10s+10} \right)_{s=0} \cdot I$$

$$V = (1) \cdot (0.3186)$$

$$v(t) = 0.3186$$

Case 2:

$$I(t) = 0.2895 \cos(10t) - 0.4513 \sin(10t)$$

Using phasor analysis

$$I = 0.2895 + j0.4513$$

$$s = j10$$

$$V = \left(\frac{s+10}{s^2+10s+10} \right)_{s=j10} \cdot I$$

$$V = (0.0055 - j0.1050) \cdot (0.2895 + j0.4513)$$

$$V = -0.0458 - 0.0329j$$

$$v(t) = -0.0458 \cos(10t) + 0.0329 \sin(10t)$$

Case 3:

$$I(t) = 0.1208 \cos(20t) - 0.2634 \sin(20t)$$

Using phasor analysis

$$I = 0.1208 + j0.2634$$

$$s = j20$$

$$V = \left(\frac{s+10}{s^2+10s+10} \right)_{s=j20} \cdot I$$

$$V = (0.0005 - j0.0510) \cdot (0.1208 + j0.2634)$$

$$V = -0.0135 + j0.0060$$

$$v(t) = -0.0135 \cos(20t) - 0.0060 \sin(20t)$$

To get the total input, add up $I(t)$ for part a), b), and c)

To get the total output, add up $V(t)$ for part a), b), and c)

$$v(t) = v_0(t) + v_{10}(t) + v_{20}(t)$$

$$v(t) = 0.3186 - 0.0458 \cos(10t) + 0.0329 \sin(10t) - 0.0135 \cos(20t) - 0.0060 \sin(20t)$$

This looks like the following:

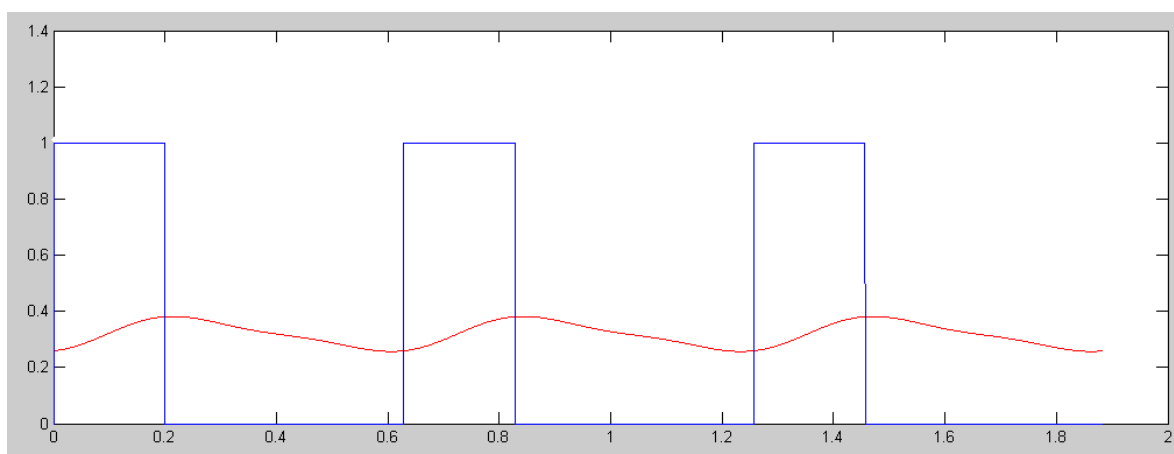


Figure 5: $I(t)$ (blue) and $V(t)$ (red) - shown for three cycles

Matlab Solution

In Matlab, this is actually a lot easier (hint: use this method for solving the homework)

First, input the function. Assume the period is $\frac{2\pi}{10}$

```
t = [0:0.0001:1]' * 2*pi/10;
x = 1 * (t < 0.2);
```

a) At $s = 0$ (DC),

```
s = 0;
a0 = mean(x);
I0 = a0
    0.3186          ( the DC component of I(t) )
G0 = ( s + 10 ) / ( s^2 + 10*s + 10 )
    1              ( the gain at s = 0 )
V0 = G0 * I0
    0.3186          ( the DC component of V(t) )
```

b) At 10 rad/sec (1st harmonic)

```
s = j*10;
a10 = 2*mean(x .* cos(10*t));
b10 = 2*mean(x .* sin(10*t));
I10 = a10 - j*b10
    0.2896 - 0.4513i  ( the phasor representation for I(t) at 10 rad/sec )
G10 = ( s + 10 ) / ( s^2 + 10*s + 10 )
    0.0055 - 0.1050i  ( the gain at s = j10 )
V10 = G10 * I10
   -0.0458 - 0.0329i  ( the phasor representation for V(t) at 10 rad/sec )
```

c) At 20 rad/sec:

```
s = j*20;

a20 = 2 * mean( x .* cos(20*t) );
b20 = 2 * mean( x .* sin(20*t) );

I20 = a20 - j*b20

    -0.1207 - 0.2634i    ( the phasor representation for I(t) at 20 rad/sec )

G20 = ( s + 10 ) / ( s^2 + 10*s + 10 )

    0.0005 - 0.0510i    ( the gain at s = j20 )

V20 = G20 * I20

    -0.0135 + 0.0060i    ( the phasor representation for V(t) at 20 rad/sec )
```

Finally, convert from phasor notation back to time domain:

```
% Start with the DC term...
V = V0;

% add in the 10 rad/sec term...
V = V + real(V10)*cos(10*t) - imag(V10)*sin(10*t);

% add in the 20 rad/sec term...
V = V + real(V20)*cos(20*t) - imag(V20)*sin(20*t);

plot(t,I,t,V)
```

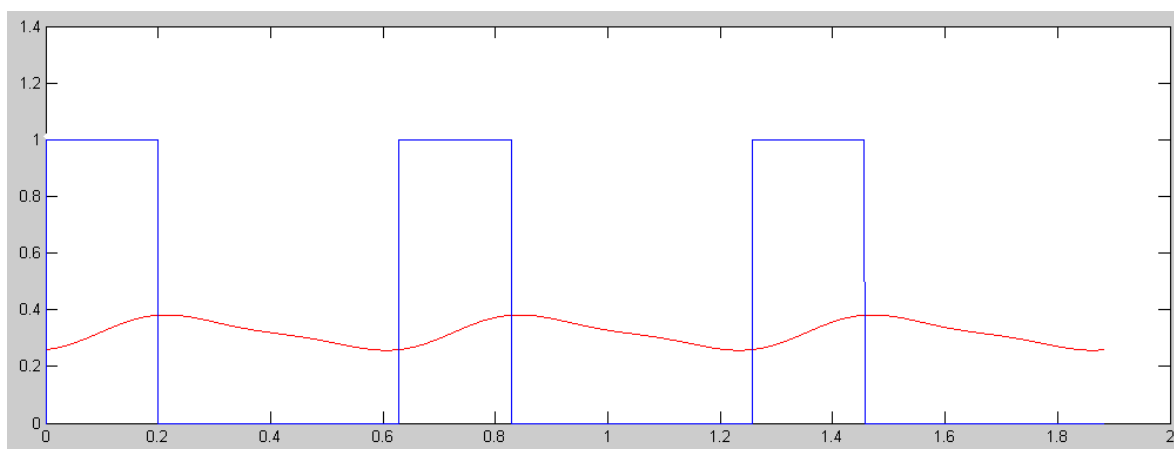


Figure 6: I(t) (blue) and V(t) (red) - shown for three cycles (same results as before)