ECE 311 Circuits 2: Phasors

Topics

- Phasors
- Representing voltages using phasors
- Representing RLC using phasors
- AC circuit analysis using phasors
- HP42 Calculator

Introduction

In the first part of Circuits I, we deal with the DC analysis of resistor circuits. When dealing DC signals, real numbers work just fine:

- Voltages can be expressed by a real number
- Currents can be expressed with real numbers, and
- Resistance's can be expressed with real numbers.

Towards the end of Circuits I and throughout Circuits II, however, we start to work with sinusoidal signals.

With sinusoids, you have two terms for each voltage and current: a sine term and a cosine term. When dealing with AC analysis, you need a number which has two components to represent these two terms. Complex numbers have that. Likewise, when dealing with AC analysis, complex numbers make the analysis much easier.

There are two ways to represent any complex number: rectangular and polar form:

$$P = a + jb = r\angle\theta$$

with the conversion being

$$a = r \cdot \cos\theta \qquad b = r \cdot \sin\theta$$

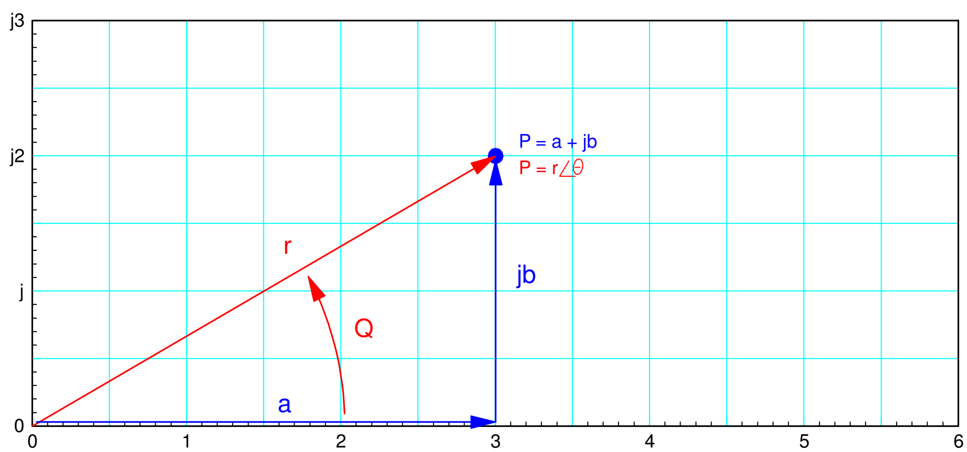

Figure 1: Complex nuumbers can be expressed in rectangular or polar form

Phasor Representation of Voltages

Euler's identity states that the complex exponential is

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

If you assume all functions are of the form of $e^{j\omega t}$, then you can represent cosine by taking the real part:

$$\cos(\omega t) = real(e^{j\omega t})$$

Likewise, the phasor (complex-number) representation for cosine is one

$$1 \leftrightarrow \cos(\omega t)$$

If you multiply by (a + jb)

$$(a+jb)e^{j\omega t} = (a+jb)(\cos(\omega t) + j\sin(\omega t))$$

$$= (a\cos(\omega t) - b\sin(\omega t)) + j(\cdots)$$

and take the real part you get the phasor representation for a generalized sine wave

$$a + jb \leftrightarrow a\cos(\omega t) - b\sin(\omega t)$$

You can also represent voltages in polar form:

$$r\angle\theta \leftrightarrow r\cos(\omega t + \theta)$$

Example: Determine x(t) for the following waveform:

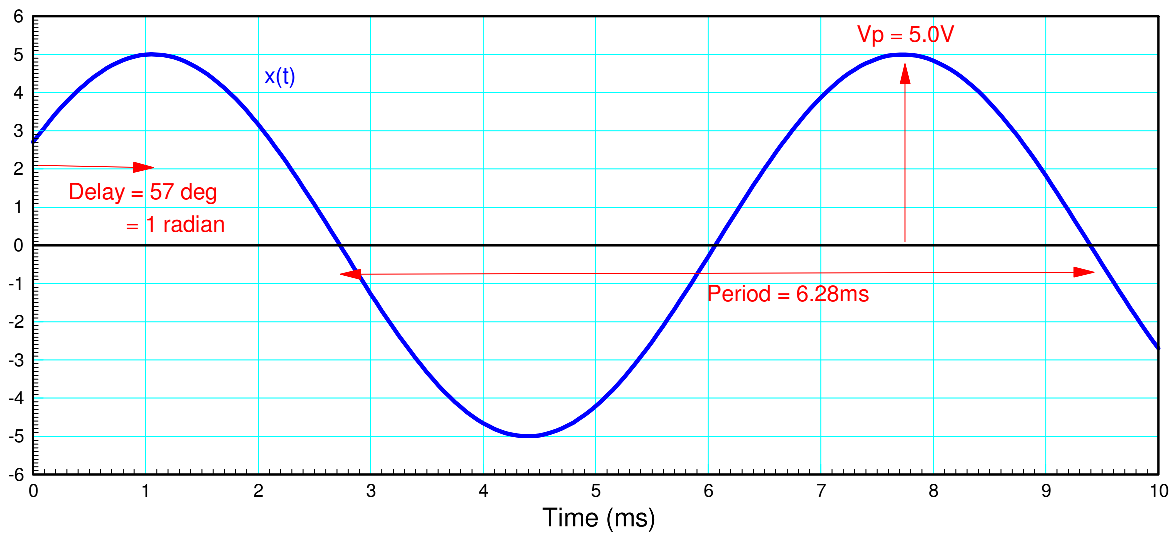

Figure 2: Typical Sinusoid: Determine x(t)

The frequency comes from the period

$$T = 6.28ms$$

$$f = \frac{1}{T} = 159.2Hz = 159.2\frac{\text{cycles}}{\text{second}}$$

$$\omega = 2\pi f = 1000\frac{\text{rad}}{\text{sec}}$$

(note that natural units are rad/sec, not Hz). The amplitude is 5.00V. The delay is 1ms, which works out to

$$\theta = -\left(\frac{\text{1ms delay}}{\text{6.28ms period}}\right)2\pi = -1.0\,\text{radian} = \text{-}57.3^0$$

meaning

$$x(t) = 5\cos(1000t - 1)$$

The phasor representation for x(t) is

$$X = 5\angle -1 = 5\angle -57.1^0$$

Doing a polar-to-rectangular conversion gives the equivalent

$$X = 2.70 - j4.21$$

which gives the other way to represent x(t)

$$x(t) = 2.70\cos(1000t) + 4.21\sin(1000t)$$

Phasor Representation for Impedance's

When dealing with AC signals, resistors, capacitors, and inductors can all be used. Phasor analysis converts each of these to a complex impedance.

Resistors: The VI relationship for a resistor is

$$V = IR$$

The phasor impedance of a resistor is R.

$$R \rightarrow R$$

Inductors: The VI relationship for an inductor is

$$V = L\frac{dI}{dt}$$

Assuming all signals are in the form of $e^{j\omega t}$, this means

$$V = L\frac{d}{dt}(e^{j\omega t}) = j\omega L\ e^{j\omega t} = j\omega L \cdot I$$

The impedance of an inductor is $j\omega L$

$$L \rightarrow j\omega L$$

Capacitors: The VI relationship for a capacitor is

$$V = \frac{1}{C}\int I\ dt$$

Assuming all signals are in the form of $e^{j\omega t}$, this means

$$V = \frac{1}{C}\int (e^{j\omega t});dt = \left(\frac{1}{j\omega C}\right)e^{j\omega t} = \left(\frac{1}{j\omega C}\right)I$$

The impedance of a capacitor is $\left(\frac{1}{j\omega C}\right)$

$$C \rightarrow \frac{1}{j\omega C}$$

ELI the ICE Man

With capacitors and inductors, the 'j' term tells you there is a phase shift (time shift) between voltage and currents for inductors and capacitors. One way to remember this is the phrase *ELI the ICE man*.

- For inductors (L), voltage leads current (ELI)
- For capacitors (C), current leads voltage (ICE).

This show up with phasors. If $\omega L = 1$ then

$$L \rightarrow j\omega L = j = 1\angle 90^0$$

If you apply a 1V sine wave at 1 rad/sec, the current and voltage will be related by

$$V = I \cdot j\omega L = I \cdot 1\angle 90^0$$

The voltage will be 90 degrees ahead of the current (Voltage leads current by 90 degrees).

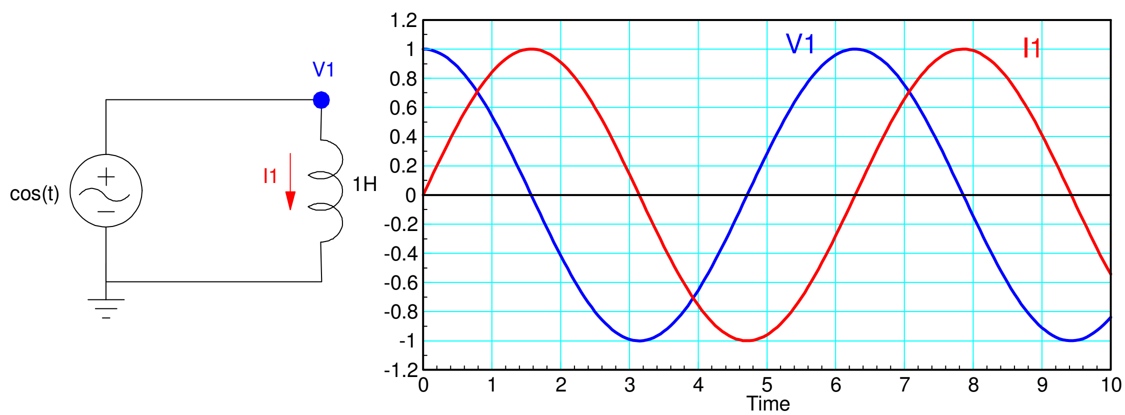

Figure 3: For inductors, voltage leads current (ELI)

For capacitors, current leads voltage. This shows up in the impedance. If $\omega C = 1$, then

$$\frac{1}{j\omega C} = -j = 1\angle -90^0$$

and

$$V = I \cdot 1\angle -90^0$$

The voltage will have a 90 degree delay from the current (ICE: current leads voltage for capacitors)

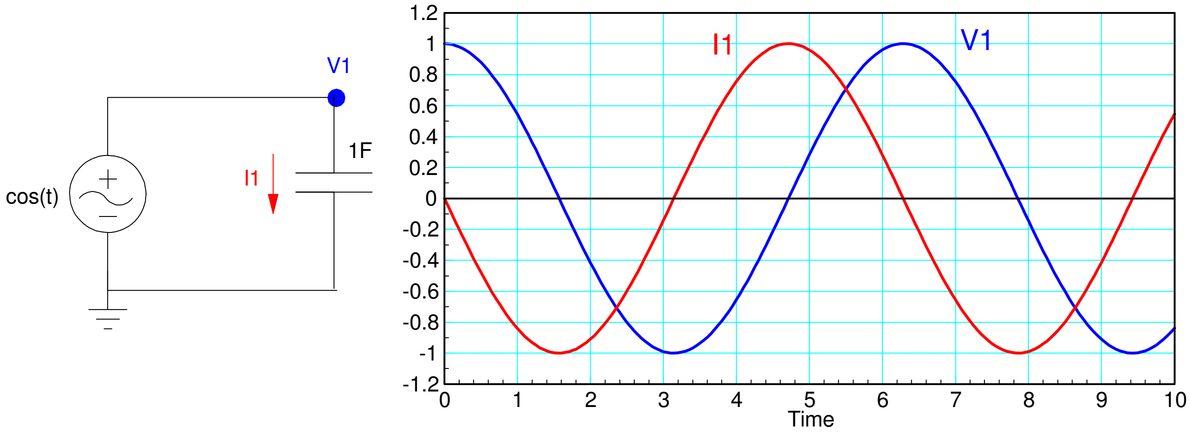

Figure 4: For capacitors, current leads votlage (ICE)

Phasor Summary

Component	Phasor Representation
V = a cos(wt) - b sin(wt) I = a cos(wt) - b sin(wt)	a + jb
R	R
L	jwL
C	1 / jwC

Table 1: Electrical components and their phasor representation

Simplification of RLC circuits

When dealing with resistor circuits, we could simplify them by adding resistors in series and parallel. The same holds with RLC circuits: we add components in series and parallel, only now while dealing with complex numbers. Note the impedance uses the term Z rather than R, since the result will be a complex number.

Example: Determine the impedance Zab.

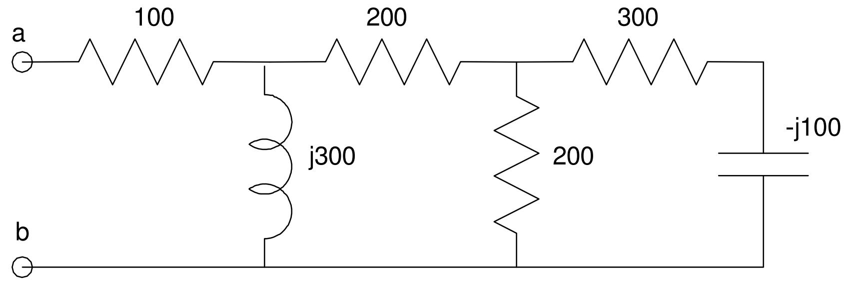

Figure 5: Determine the impedance between a and b (it will be a complex number)

Solution: Do like we did before, but now with complex numbers.

-j100 and +300 are in series

-j100 + 300 = 300 - j100

This is in parallel with 200

$$(200)||(300-j100) = \left(\frac{1}{200}+\frac{1}{300-j100}\right)^{-1} = 123.07-j15.38$$

Which is in series with 200

$$(200)+(123.07-j15.38) = 323.07-j15.38$$

Which is in parallel with +j300

$$(j300)||(323.08-j15.38) = 156.85+j161.83$$

which is in series with 100

$$(100)+(156.85+j161.83) = 256.85+j161.83$$

Answer:

$$Z_{ab} = 256.85+j161.83$$

Solving in Matlab

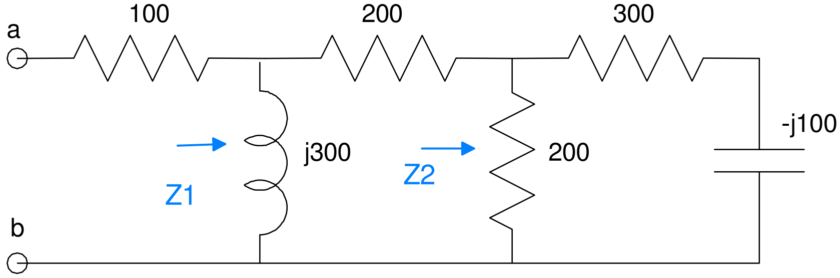

Figure 6: Collapse the circuit from right to left to find the impedance Zab

```
>> Z2 = 1 / (1/200 + 1/(300 - j*100) )

Z2 =  1.2308e+002 -1.5385e+001i

>> Z1 = 1 / (1/(j*300) + 1/(200 + Z2) )

Z1 =  1.5685e+002 +1.6183e+002i

>> Zab = 100 + Z1
```

`Zab =  2.5685e+002 +1.6183e+002i`

Solving with an HP42

```
300
enter
-100
compelx
1/X
200
1/X
+
1/X           Z2 = 123.0769 - j15.3846
200
+
1/X
0
enter
300
complex
1/X
+
1/X           Z1 = 156.8465 + j161.8257
100
+             Zab = 245.8456 + j161.8257
```

Solving in CircuitLab

Zab tells you that current and voltage are related by

$$V = I \cdot Z_{ab}$$

$$V = I \cdot (245.8456 + j161.8257)$$

This shows up better in polar form:

$$V = I \cdot (294.32\angle 33.35^0)$$

Translation:

- The voltage will be 294.32 times larger than the current
- Voltage leads current by 33.35 degrees

To make the math easier, let the frequency be 1 rad/sec (0.1591 rad/sec). Then a circuit with these impedances is (note: to have -j100 Ohms for the capacitor, C = 0.01F due to Z = 1/(jwC)

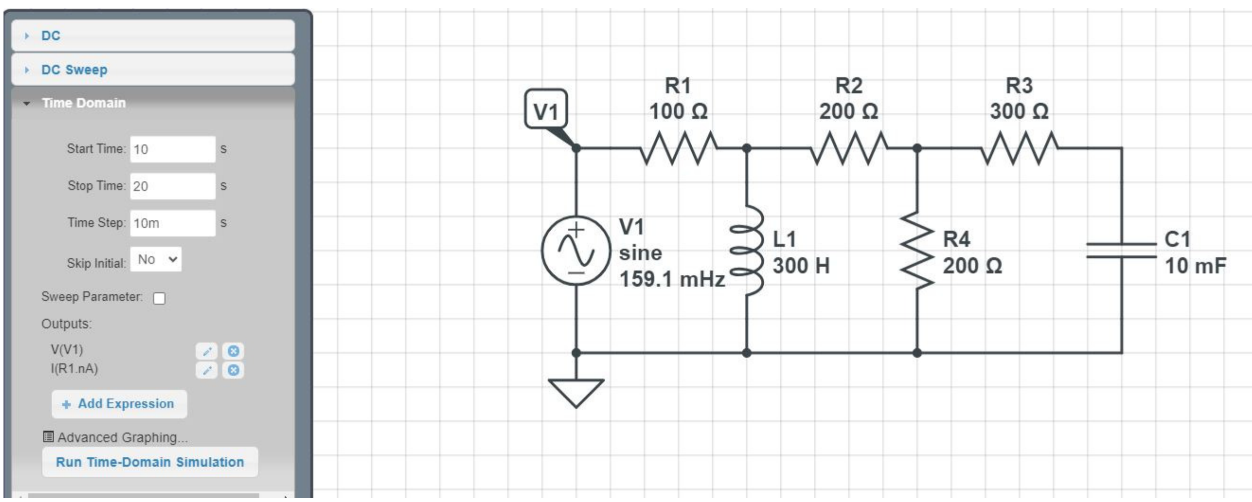

Figure 7: CircuitLab simulation with a 1 rad/sec input
Note: C = 1/100F for Zc = -j100 Ohms

If you plot the voltage going into the circuit as well as the current going in (click on the left side of R1), you see

- The peak voltage is 294 times the peak current
- The peak voltage is 33 degrees before the peak current

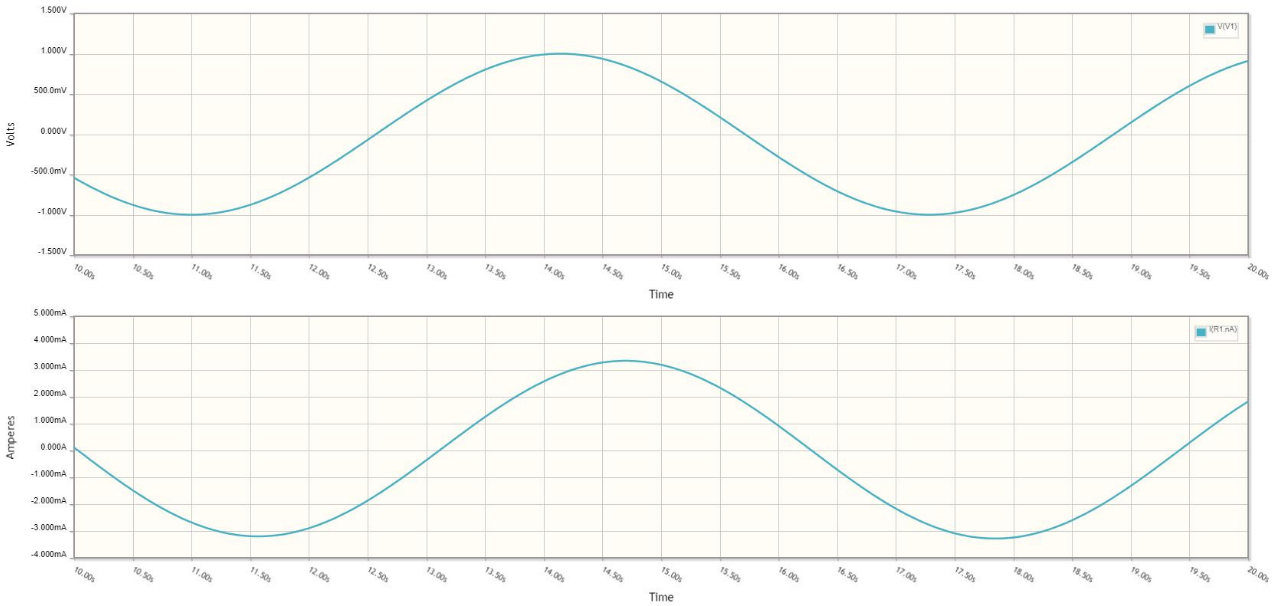

Figure 8: Voltage and Current to the circuit
max(V) = 294 * max(I). Voltage leads current by +33 degrees.

Example 2: Determine Zab

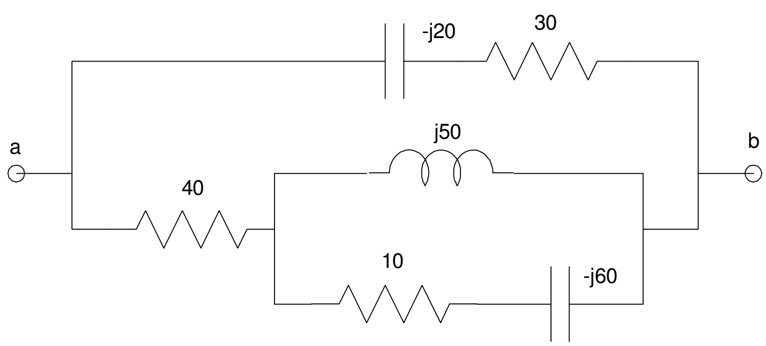

Figure 9: Determine the impedance between a) and b) (it will be a complex number)

Solution

$$(10) + (-j60) = 10 - j60$$

$$(10 - j60)||(j50) = 125.00 + j175.00$$

$$(125.00 + j175.00) + (40) = 165.00 + j175.00$$

$$(165.00 + j175.00)||(30 - j20) = 31.42 - j14.98$$

answer:

$$Z_{ab} = 31.42 - j14.98$$

Solve in Matlab:

```
>> Z1 = 10 - j*60

Z1 =  10.0000 -60.0000i

>> Z2 = 1 / (1/Z1 + 1/(j*50))

Z2 =  1.2500e+002 +1.7500e+002i

>> Z3 = Z2 + 40

Z3 =  1.6500e+002 +1.7500e+002i

>> Z4 = 1/( 1/Z3 + 1/(30-j*20) )
```

Z4 = 31.4263 -14.9799i

Solve using an HP42

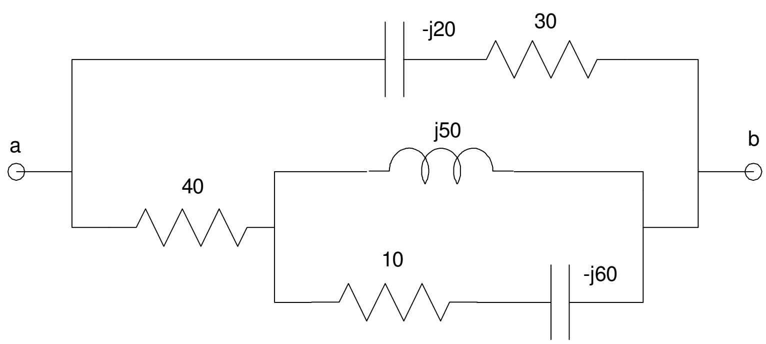

Figure 10: Find the impedance using an HP42 calculator (Free42 app on your cell phone)

```
10
enter
-60
complex
1/X
0
enter
50
complex
1/X
+
1/X                    Z2 = 125 + j175
40
+                      Z3 = 165 + j175
1/X
30
enter
-20
complex
1/X
+
1/X                    Zab = 31.4263 - j14.9799
```

Solve with CircuitLab:

Same trick as before

- Let the input be a 1V, 1 rad/sec sine wave (0.1591 Hz)
- Let L be the impedances of the inductors (jwL)
- Let C be the inverse of the impedance of the capacitors (1/jwC)

Again, polar form is easier to see what should happen. Voltage and current are related by

$$V = I \cdot (31.4263 - j14.9799)$$

or in polar form

$$V = I \cdot (34.81\angle -25.49^0)$$

This means

- Voltage should be 34.81 times larger than current
- Voltage should lag behind current by 25.49 degrees.

This is what you see in CircuitLab

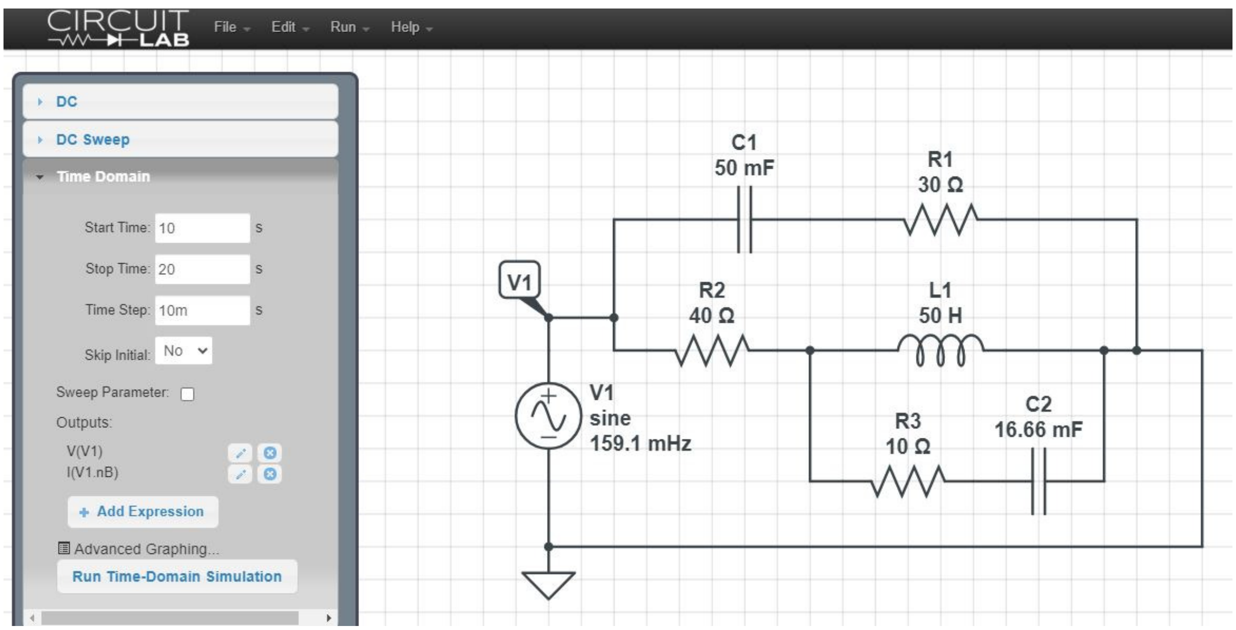

Figure 11: CircuitLab simulation to check the impedance

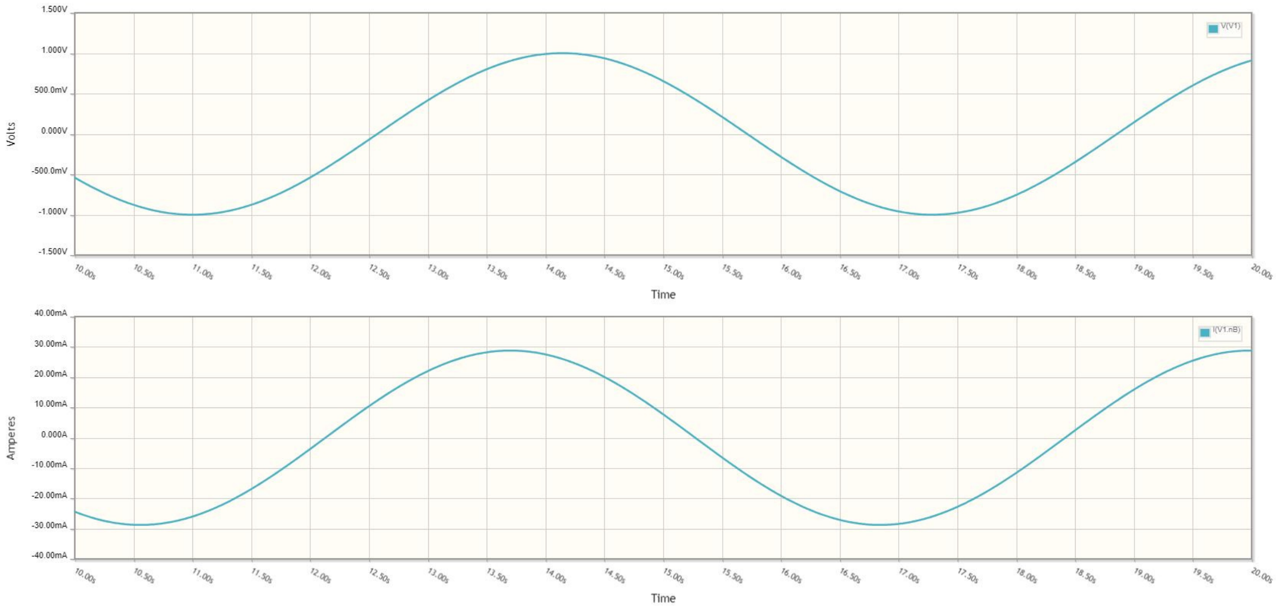

Figure 12: Voltage & current going to the RLC network
Voltage is 34.81 times larger than current, delayed by 25.49 degrees

Circuit Analysis with Phasors

Everything we did at DC still works for AC analysis, only now with complex numbers,.

Example 1: RC Circuit Determine y(t) for the following circuit where V0 is a 10V peak, 100Hz sine wave:

$$V_0 = 10 \sin(628t)$$

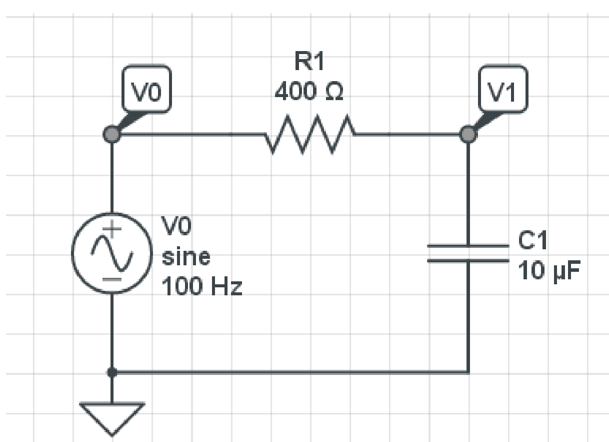

Figure 13: Single-stage RC circuit. Find V1 using phasors.

Step 1: Replace the capacitor with its complex impedance. Since the input is 628 rad/sec, that's the frequency you care about

$$\omega = 628 \quad \text{rad/sec}$$

$$Z_c = \frac{1}{j\omega C} = -j159\Omega$$

$$X = 0 - j10$$

Step 2: Solve just like you did with a DC circuit, only with complex numbers

$$Y = \left(\frac{-j159}{-j159+400}\right)(0 - j10)$$

$$Y = -3.436 - j1.368$$

$$y(t) = -3.436\cos(628t) + 1.368\sin(628t)$$

In polar form

$$Y = 3.694\angle -158.3^0$$

meaning

$$y(t) = 3.694\cos(628t - 158.3^0)$$

Both forms are valid answers.

In CircuitLab, input the circuit using drag and drop

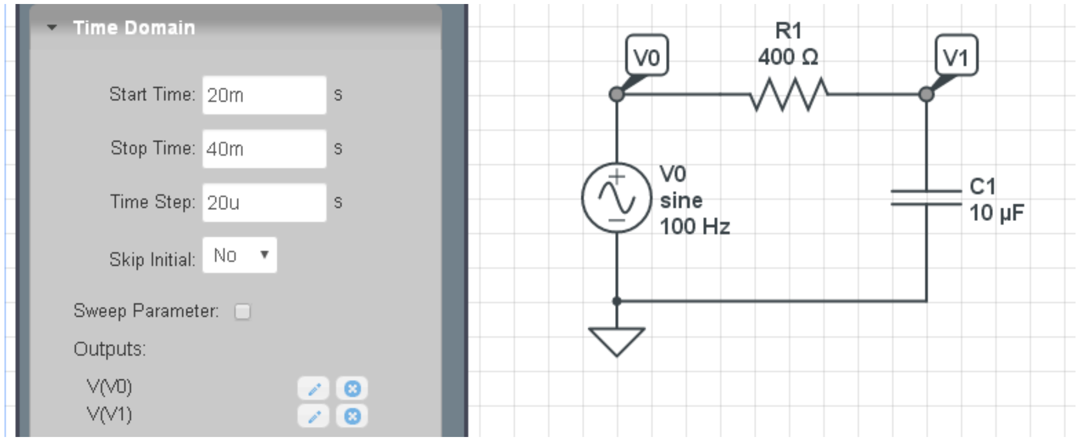

Figure 14: Time-domain simulation in CircuitLab to verify calculations

Run a time-domain response for 20ms (2 cycles). This results in a simulated input and output waveform:

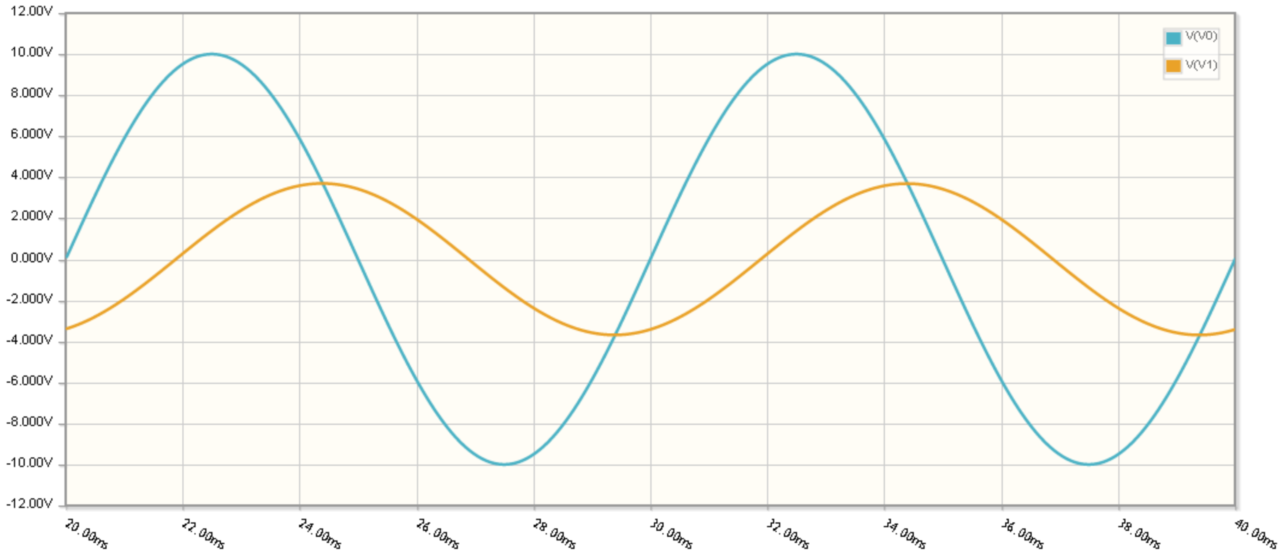

Figure 15: Time-domain simulation results. Input (blue) and Output (orange)

Note from the CircuitLab plot, matches our calculations:

- The peak for V1 is 3.694V
- The peak for V1 is delayed from t = 0 (the zero crossing for V0) by 4.5ms. This works out to

$$\theta = -\left(\frac{\text{delay (ms)}}{\text{period (ms)}}\right)360^0 = -\left(\frac{4.5ms}{10ms}\right)360^0 = -162^0$$

which matches our computations of -158.2 degrees

Example 2: 3-Stage RC Circuit

Find the voltages for the following circuit

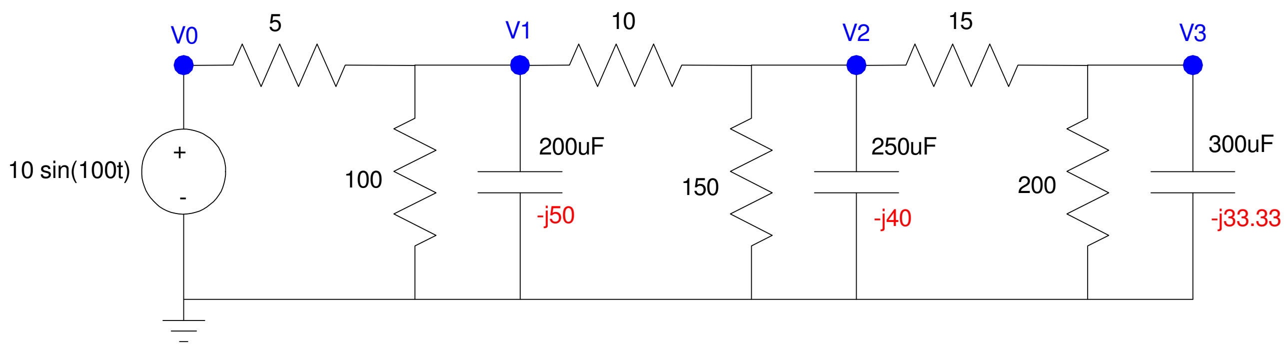

Figure 16: Determine the voltages for an RC circuit using pahsors.

Step 1: Change the capacitors to their complex impedance (shown in red)

$$\omega = 100$$

0.01F: $Z_c = \frac{1}{j\omega C} = -j50\Omega$

0.02F: $Z_c = \frac{1}{j\omega C} = -j40\Omega$

0.03F: $Z_c = \frac{1}{j\omega C} = -j33.33\Omega$

Step 2: Write N equations for N unknowns

V0: $V_0 = 0 - j10$

V1: $\left(\frac{V_1 - V_0}{5}\right) + \left(\frac{V_1}{100}\right) + \left(\frac{V_1}{-j50}\right) + \left(\frac{V_1 - V_2}{10}\right) = 0$

V2: $\left(\frac{V_2 - V_1}{10}\right) + \left(\frac{V_2}{150}\right) + \left(\frac{V_2}{-j40}\right) + \left(\frac{V_2 - V_3}{15}\right) = 0$

V3: $\left(\frac{V_3 - V_2}{15}\right) + \left(\frac{V_3}{200}\right) + \left(\frac{V_3}{-j33.33}\right) = 0$

Step 3: Solve. First, group terms

$$V_0 = -j10$$

$$-\left(\frac{1}{5}\right) V_0 + \left(\frac{1}{5} + \frac{1}{100} + \frac{1}{-j50} + \frac{1}{10}\right) V_1 + \left(\frac{-1}{10}\right) V_2 = 0$$

$$\left(\frac{-1}{10}\right) V_1 + \left(\frac{1}{10} + \frac{1}{150} + \frac{1}{-j40} + \frac{1}{15}\right) V_2 + \left(\frac{-1}{15}\right) V_3 = 0$$

$$\left(\frac{-1}{15}\right) V_2 + \left(\frac{1}{15} + \frac{1}{200} + \frac{1}{-j33.33}\right) V_3 = 0$$

Place in matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \left(\frac{-1}{5}\right) & \left(\frac{1}{5}+\frac{1}{100}+\frac{1}{-j50}+\frac{1}{10}\right) & \left(\frac{-1}{10}\right) & 0 \\ 0 & \left(\frac{-1}{10}\right) & \left(\frac{1}{10}+\frac{1}{150}+\frac{1}{-j40}+\frac{1}{15}\right) & \left(\frac{-1}{15}\right) \\ 0 & 0 & \left(\frac{-1}{15}\right) & \left(\frac{1}{15}+\frac{1}{200}+\frac{1}{-j33.33}\right) \end{bmatrix} \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} -j10 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Put into MATLAB and solve

```
a1 = [1,0,0,0];
a2 = [-1/5,1/5+1/100+1/(-j*50)+1/10,-1/10,0];
a3 = [0,-1/10,1/10+1/150+1/(-j*40)+1/15,-1/15];
a4 = [0,0,-1/15,1/15+1/200+1/(-j*33.33)];
A = [a1;a2;a3;a4]

   1.0000                   0                   0                   0
  -0.2000             0.3100 + 0.0200i  -0.1000                   0
        0            -0.1000              0.1733 + 0.0250i  -0.0667
        0                   0            -0.0667              0.0717 + 0.0300i

B = [-j*10;0;0;0];
V = inv(A)*B

V0        0 -10.0000i
V1  -1.6314 - 8.0724i
V2  -3.4430 - 5.3506i
V3  -4.4982 - 3.0942i
```

meaning

$$V_0 = 10\sin(100t)$$

$$V_1 = -1.6314\cos(100t) + 8.0742\sin(100t)$$

$$V_2 = -3.4430\cos(100t) + 5.5306\sin(100t)$$

$$V_3 = -4.4982\cos(100t) - 3.0942\sin(100t)$$

The magnitude of each voltage is:

```
abs(V)

   10.0000
    8.2356
    6.3627
    5.4596
```

In CircuitLab, you can check this answer. First, input the circuit then run a Time Domain simulation:

CircuitLab Simulation

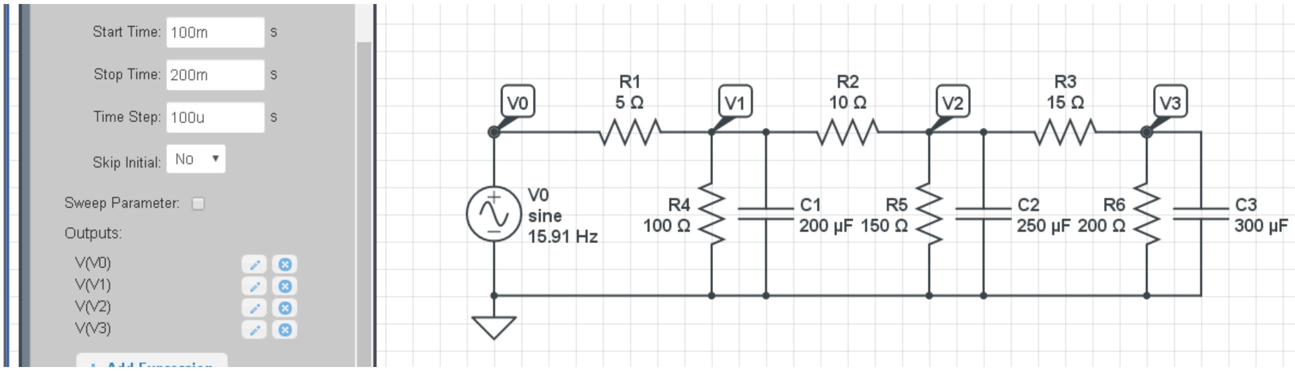

Figure 17: Time-domain simulation in CircuitLab to check calculations

The results is as follows: the peak voltage should match with the magnitude of V0 .. V4

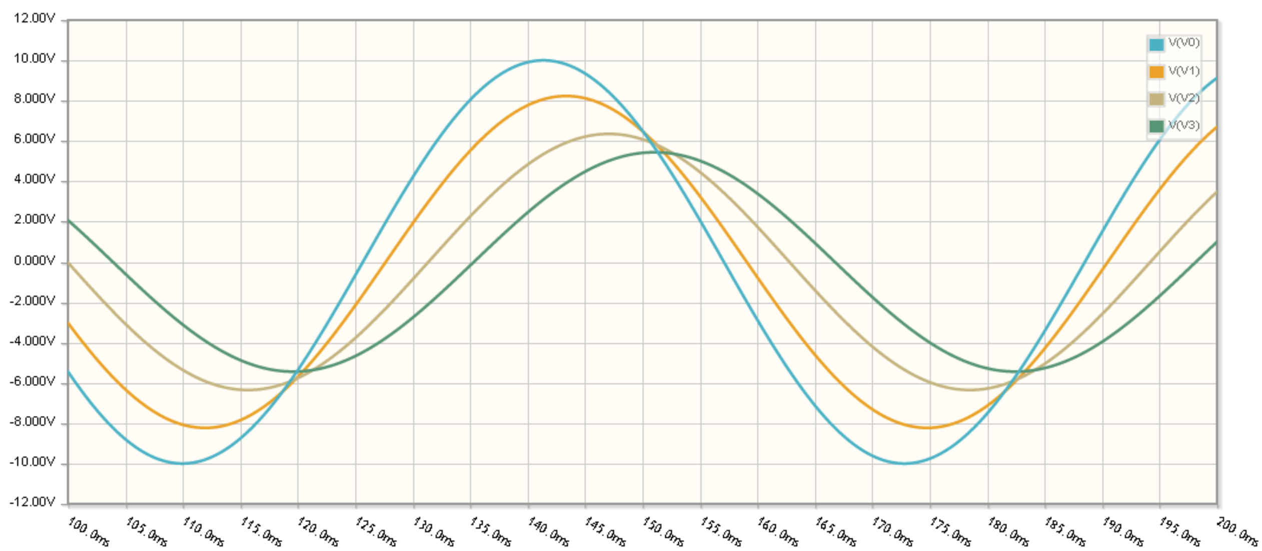

Figure 18: Transient Response from CircuitLab.

	|Vin|	|V1|	|V2|	|V3|
Calculated	10.0000V	8.2356V	6.3627V	5.4596V
CircuitLab	10.00V	8.231V	6.348V	5.435V

Table 2: Magnitudes of calculated voltages should match peak voltage from CircuitLab simulation

Summary

Real numbers work well when analyzing DC circuits

Complex numbers work well when analyzing AC circuits

Everything that we did with DC circuits with AC circuits - only you wind up with complex numbers

For voltages

- The real part represents the cosine term
- The complex part is represents the minus-sine term

For impedances

- Resistors are real (R)
- Inductors are +jX (Z = jwL)
- Capacitors are -jX (Z = 1/(jwC))