

Inductors and the Wave Equation

Topics

- Inductors
- Wave Equation
- Reflection Coefficients
- Animation in Matlab

Inductors

Capacitors deal with voltage and store energy in an electric field. Inductors, on the other hand, deal with current and store energy in a magnetic field.

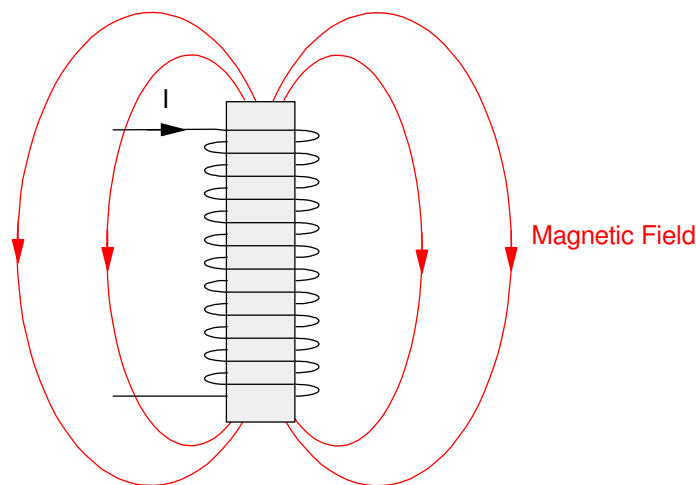


Figure 1: Inductors: Coils of wire produce a magnetic field, which in turns produces an inductor

Inductance is measured in units of Henries. The way you build an inductor is

- You create a coil like the one shown above, or
- You just carry current with a wire.

Example 1: Determine the inductance and resistance of an inductor like the one shown above. Assume

- 100 windings of 36 gage wire copper wire (1.38 Ohms / meter)¹
- Cross sectional area = 25mm²
- Length = 10mm
- Air core (relative permeability of 1.000)

Solution: From Electronics Tutorials²

$$L = \mu N^2 A / l$$

¹ https://www.engineeringtoolbox.com/copper-wire-d_1429.html

² <https://www.electronics-tutorials.ws/inductor/inductor.html>

$$L = \left(4\pi \cdot 10^{-7} \frac{H}{m}\right) (100)^2 (25 \cdot 10^{-6} m^2) / (0.01 m)$$

$$L = 31.42 \mu H$$

The resistance is

$$R = (100 \text{ windings}) \left(0.02 \frac{m}{\text{winding}}\right) (1.38 \frac{\Omega}{m})$$

$$R = 2.77 \Omega$$

This is typical for inductors:

- The inductance is small: 0.000 0314 Henries
- The inductor has a sizable resistance: 2.77 Ohms

If you place iron in the core, (relative permeability of iron is 800), the inductance goes up 800x (!)

$$L = 25.1 mH$$

This is also typical for inductors: they often have iron cores making them large and heavy.

Example 2: Determine the inductance of a copper transmission line:

- Length = 1km
- radius = 1cm
- frequency = 60Hz

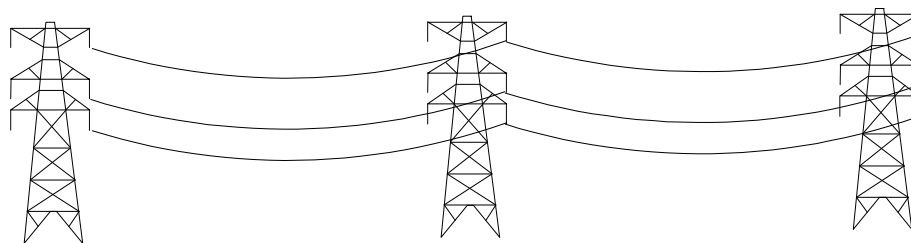


Figure 2: Long transmission lines have a magnetic field, and hence inductance

From Wikipedia³

$$L = \frac{\mu_0}{2\pi} l (A - B + C)$$

$$A = \ln \left(\frac{l}{r} + \sqrt{\left(\frac{l}{r}\right)^2 + 1} \right) = 12.20$$

$$B = \frac{1}{\frac{r}{l} + \sqrt{1 + \left(\frac{r}{l}\right)^2}} \approx 1$$

³ https://en.wikipedia.org/wiki/Inductor#Inductance_formulas

$$C = \frac{1}{4 + r\sqrt{\frac{2}{\rho}\omega\mu}} = 0.1569$$

giving

$$L = 2.30 \frac{mH}{km}$$

The inductance isn't a lot, but when a transmission line travels thousands of kilometers, it adds up.

VI Characteristics of Inductors

Inductors store energy in the magnetic field as

$$E = \frac{1}{2}LI^2 \quad \text{Joules}$$

The power absorbed is the derivative of energy

$$P = \frac{dE}{dt} \quad \text{Watts}$$

or

$$P = VI = LI \frac{dI}{dt}$$

Canceling terms gives the VI relationship for inductors:

$$V = L \frac{dI}{dt}$$

Inductors act as differentiators.

Example: The current through a 1H inductor is shown below in red. Sketch the voltage.

Solution: Shown in blue. The derivative (times one) is the voltage.

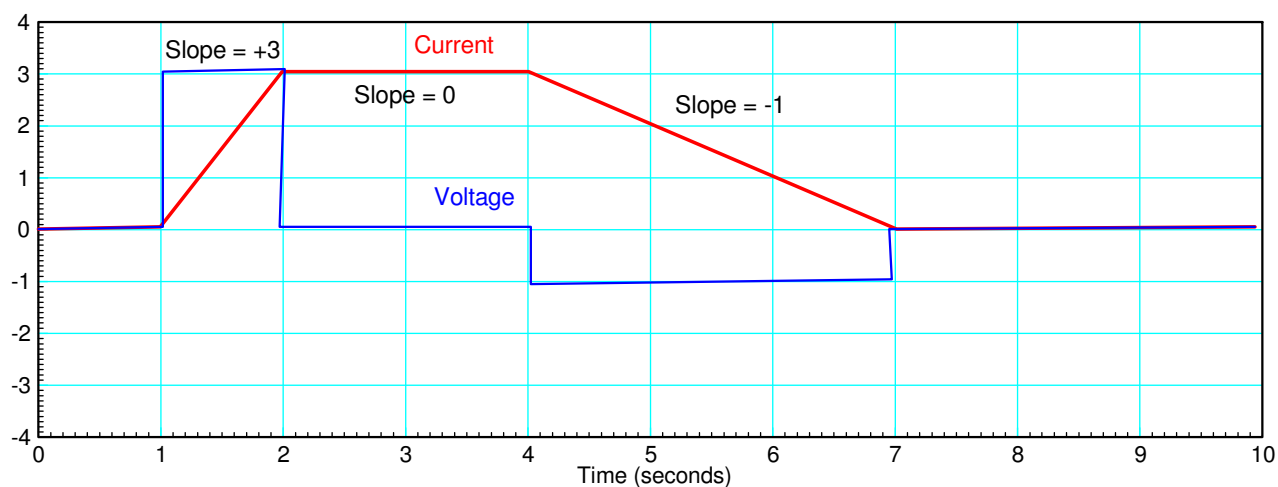


Figure 3: The voltage across an inductor is proportional to the derivative of the current

Heat Equation (recap)

If you have a single capacitor in a circuit such as follows

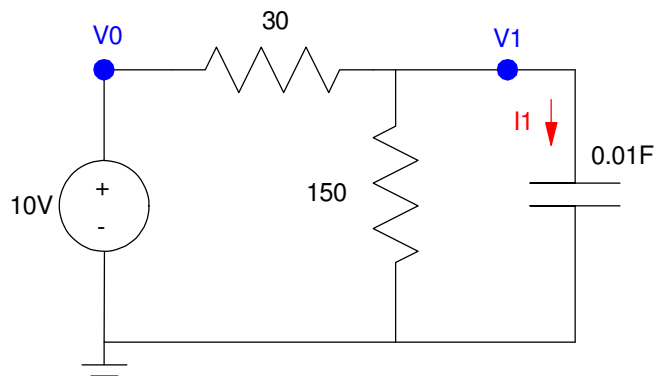


Figure 4: RC circuit with a single capacitor

you get a 1st-order differential equation.

$$I_1 = C\dot{V}_1 = \left(\frac{V_0 - V_1}{30} \right) + \left(\frac{0 - V_1}{150} \right)$$

$$\dot{V}_1 = -4V_1 + 3.33V_0$$

This models

- The charging of a single capacitor in a circuit, or
- The heating of a single element, such as a pan on a stove.

If you cascade N such RC elements, you get the heat equation where the voltage at node i is:

$$\dot{V}_i = aV_{i-1} - bV_i + aV_{i+1}$$

where V represents the voltage of an RC circuit or the temperature along a metal rod.

Wave Equation (Mass-Spring)

A similar but very different differential equation is the *Wave Equation*. This describes systems of the form

$$\ddot{V}_i = aV_{i-1} - bV_i + aV_{i+1}$$

Note that this is now a set of coupled *second-order* differential equations. It looks like an innocent enough change, but this small change makes a *large* difference in how the system behaves.

To see how the wave equation comes about, let's look at a mass-spring system.

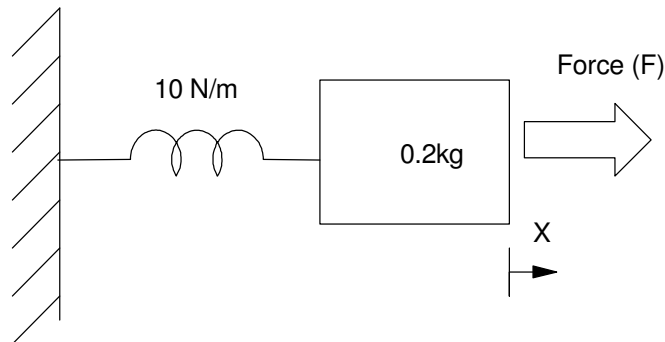


Figure 5: Mass-Spring System: described by a second-order differential equation

From

$$\text{Force} = \text{Mass} * \text{Acceleration}$$

the differential equation which describes this system is

$$F - Kx = M\ddot{x}$$

or

$$\ddot{x} = -50x + 5F$$

If you cascade N mass-spring systems together

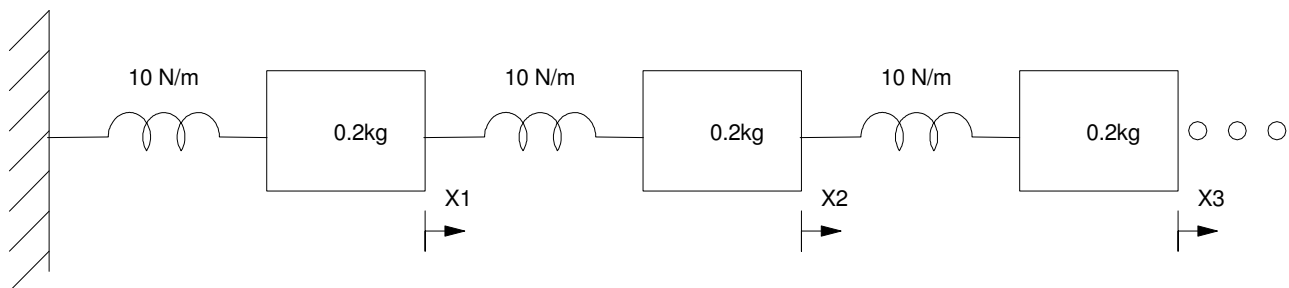


Figure 6: N-stage mass-spring system, described by N second-order differential equations

you get at mass #2

$$F_2 = m\ddot{x}_2$$

$$10(x_1 - x_2) + 10(x_3 - x_2) = 0.2\ddot{x}_2$$

or

$$\ddot{x}_2 = 10x_1 - 20x_2 + 10x_3$$

The same pattern holds for all N nodes. These coupled second-order differential equations are called *the wave equation*.

This is an important equation as it describes

- A guitar string
- Cars stuck in traffic
- Buildings swaying during an earthquake or a wind storm,
- Transmission lines galloping on windy days, and
- Bridges oscillating during wind storms.

Wave Equation (LC Circuits)

You also get similar differential equations with LC circuits. For example, find the differential equation which describes the following circuit:

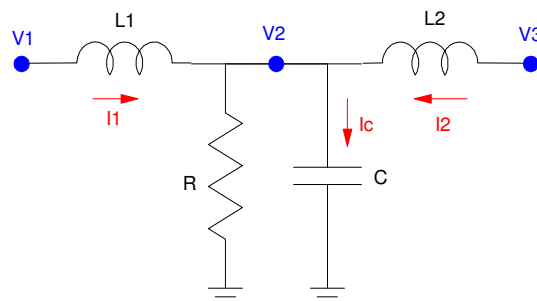


Figure 7: Single element of an RLC filter (modeling a transmission line)

This models several electrical systems

- For long transmission lines, L comes from long (1000km+) wires and C comes from the transmission lines forming a parallel plate capacitor with the ground.
- For circuit boards, L models the inductance of a wire and C models the capacitance from the circuit board trace to the ground plane.

With three elements, there are three coupled first-order differential equations to describe this circuit:

$$L_1 \dot{I}_1 = V_1 - V_2$$

$$L_2 \dot{I}_2 = V_3 - V_2$$

$$C \dot{V}_2 = I_1 + I_2 - \frac{V_2}{R}$$

Differentiating

$$C \ddot{V}_2 = \dot{I}_1 + \dot{I}_2 - \frac{\dot{V}_2}{R}$$

Substituting the inductor equation

$$C\ddot{V}_2 = \left(\frac{V_1 - V_2}{L_1}\right) + \left(\frac{V_3 - V_2}{L_2}\right) - \frac{\dot{V}_2}{R}$$

or

$$C\ddot{V}_2 = \left(\frac{1}{L_1}\right)V_1 - \left(\frac{1}{L_1} + \frac{1}{L_2}\right)V_2 + \left(\frac{1}{L_2}\right)V_3 - \left(\frac{1}{R}\right)\dot{V}_2$$

Like the mass-spring system, the voltage at node 2 (or node i) is described by a 2nd-order differential equation.

Dynamic Response of the Wave Equation

To see how the wave equation behaves, let's use Matlab to simulate an LC circuit. To start with, let's use a 3-stage LC filter:

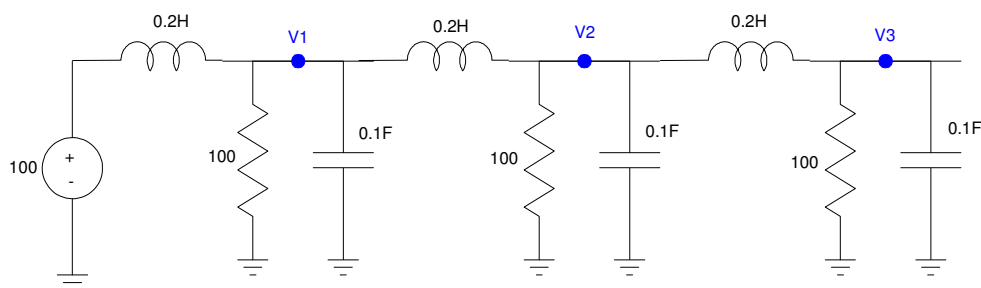


Figure 8: 3-Stage RLC filter

From before, the coupled 2nd-order differential equations for this circuit will be:

$$C\ddot{V}_i = \left(\frac{1}{L_i}\right)V_{i-1} - \left(\frac{1}{L_i} + \frac{1}{L_{i+1}}\right)V_i + \left(\frac{1}{L_{i+1}}\right)V_{i+1} - \left(\frac{1}{R}\right)\dot{V}_i$$

With all R, L, and C being the same, this simplifies to:

$$\ddot{V}_1 = \left(\frac{1}{LC}\right)V_0 - \left(\frac{2}{LC}\right)V_1 + \left(\frac{1}{LC}\right)V_2 - \left(\frac{1}{RC}\right)\dot{V}_1$$

$$\ddot{V}_2 = \left(\frac{1}{LC}\right)V_1 - \left(\frac{2}{LC}\right)V_2 + \left(\frac{1}{LC}\right)V_3 - \left(\frac{1}{RC}\right)\dot{V}_2$$

$$\ddot{V}_3 = \left(\frac{1}{LC}\right)V_2 - \left(\frac{1}{LC}\right)V_3 - \left(\frac{1}{RC}\right)\dot{V}_3$$

or

$$\ddot{V}_1 = 50V_0 - 100V_1 + 50V_2 - 0.1\dot{V}_1$$

$$\ddot{V}_2 = 50V_1 - 100V_2 + 50V_3 - 0.1\dot{V}_2$$

$$\ddot{V}_3 = 50V_2 - 50V_3 - 0.1\dot{V}_3$$

In MATLAB

```
V0 = 100;  
V1 = 0;  
V2 = 0;  
V3 = 0;  
  
dV1 = 0;  
dV2 = 0;  
dV3 = 0;  
  
V = [];  
  
dt = 0.01;  
  
for i=1:300  
  
    ddV1 = 50*V0 - 100*V1 + 50*V2 - 0.01*dV1;  
    ddV2 = 50*V1 - 100*V2 + 50*V3 - 0.01*dV2;  
    ddV3 = 50*V2 - 50*V3 - 0.01*dV3;  
  
    dV1 = dV1 + ddV1*dt;  
    dV2 = dV2 + ddV2*dt;  
    dV3 = dV3 + ddV3*dt;  
  
    V1 = V1 + dV1*dt;  
    V2 = V2 + dV2*dt;  
    V3 = V3 + dV3*dt;  
  
    V = [V; V1, V2, V3];  
  
end  
  
t = [1:300]' * dt;  
  
plot(t,V);  
xlabel('Time (seconds)');  
ylabel('Voltage');
```

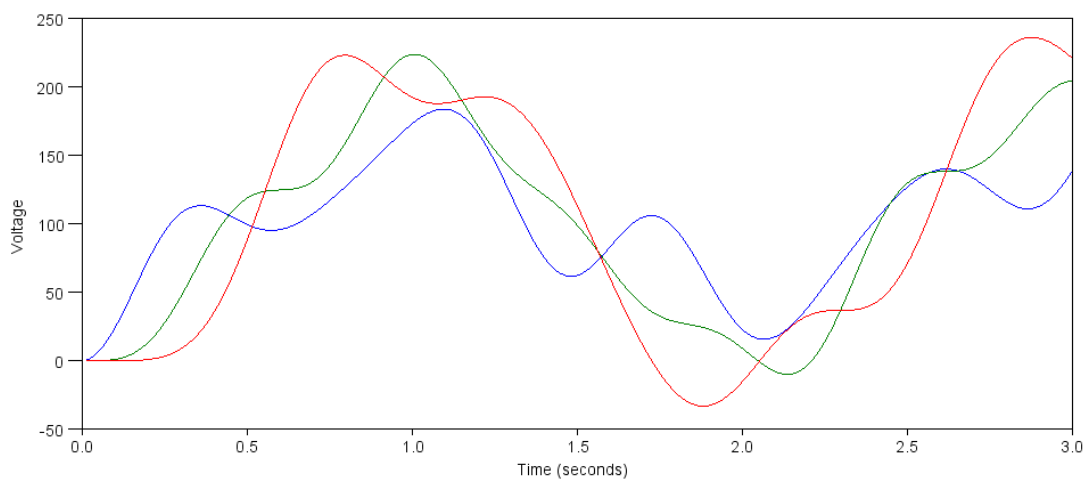


Figure 9: Voltages at Each Node for Three Seconds

It's a little more fun to watch the voltages as the simulation runs

```
V0 = 100;  
V1 = 0;  
V2 = 0;  
V3 = 0;  
  
dV1 = 0;  
dV2 = 0;  
dV3 = 0;  
  
V = [];  
  
t = 0;  
dt = 0.01;  
  
while(t < 100)  
  
    ddV1 = 50*V0 - 100*V1 + 50*V2 - 0.01*dV1;  
    ddV2 = 50*V1 - 100*V2 + 50*V3 - 0.01*dV2;  
    ddV3 = 50*V2 - 50*V3 - 0.01*dV3;  
  
    dV1 = dV1 + ddV1*dt;  
    dV2 = dV2 + ddV2*dt;  
    dV3 = dV3 + ddV3*dt;  
  
    V1 = V1 + dV1*dt;  
    V2 = V2 + dV2*dt;  
    V3 = V3 + dV3*dt;  
  
    plot([0,1,2,3],[V0,V1,V2,V3],'.-');  
    ylim([-300,300]);  
    pause(0.01);  
  
    V = [V; V1, V2, V3];  
  
end
```

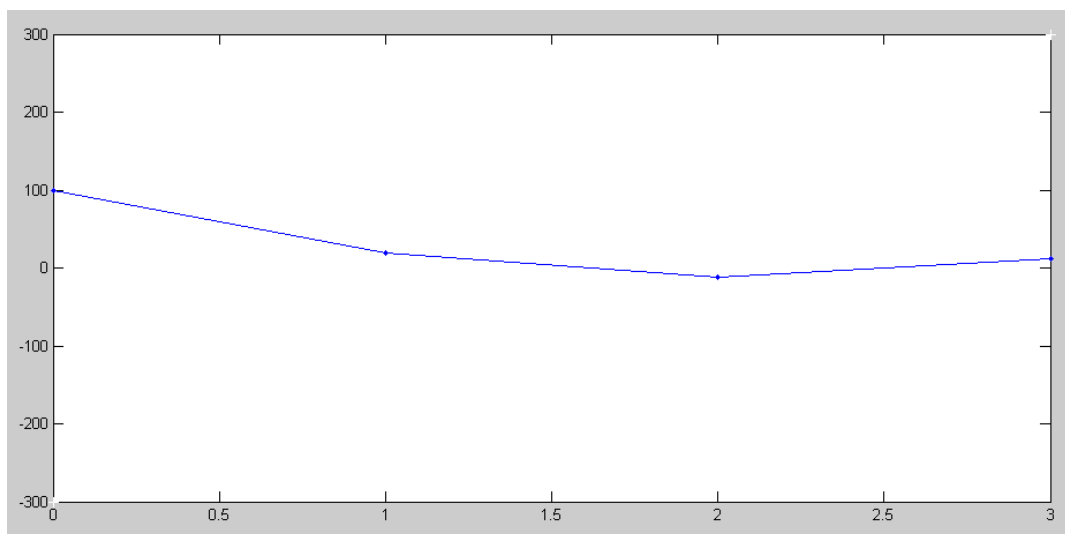


Figure 10: Node voltages at t = 10 seconds for a 3-Element Model

30-Stage Transmission Line:

A better model would increase the number of nodes. Let's use 30 nodes. Here, the equations remain the same for the first 29 nodes:

$$\ddot{V}_1 = 50V_0 - 100V_1 + 50V_2 - 0.01\dot{V}_1$$

$$\vdots$$

$$\ddot{V}_{29} = 50V_{28} - 100V_{29} + 50V_{30} - 0.01\dot{V}_{29}$$

The last node is a little different since there is only one other node connected to it

$$\ddot{V}_{30} = 50V_{29} - 50V_{30} - 0.01\dot{V}_{30}$$

In Matlab:

```
V = zeros(30,1);
dV = zeros(30,1);
t = 0;
dt = 0.01;
while(t < 1000)
    if (t < 1.5) V0 = 100* ( sin(2*t).^2);
    else V0 = 0;
    end
    ddV(1) = 50*V0 - 100*V(1) + 50*V(2) - 0.01*dV(1);
    for i=2:29
        ddV(i) = 50*V(i-1) - 100*V(i) + 50*V(i+1) - 0.01*dV(i);
    end
    ddV(30) = 50*V(29) - 50*V(30) - 0.01*dV(30);
    for i=1:30
        dV(i) = dV(i) + ddV(i)*dt;
        V(i) = V(i) + dV(i)*dt;
    end
    t = t + dt;

    plot([0:30],[V0;V],'.-');
    ylim([-150,150]);
    pause(0.01);
end
```

This is where you can see why this is called *the wave equation*. If you vary the voltage on the left, a wave is launched. This wave then travels down the transmission line until it hits the right endpoint.

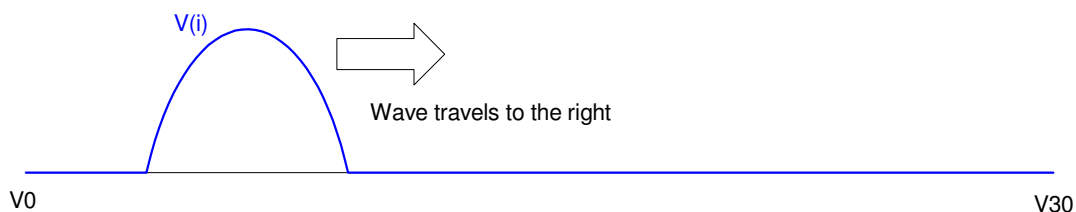


Figure 11: When V_0 changes, it launches a wave which travels to the right

Reflection Coefficients:

Note that when the wave hits the right endpoint, it reflects back with an amplitude of double the input (+200). The reflection coefficient is a way to model this:

$$\Gamma = \frac{Z_1 - Z_0}{Z_1 + Z_0}$$

where

- Z_0 is the impedance of the transmission line and
- Z_1 is the impedance of the load (R_{20} in this case)

$\Gamma = +1$: When R_{20} is large (100 Ohms here), the term Z_1 dominates and the reflection is almost +1. What this looks like is

- An infinite transmission line with a wave traveling to the right
- A second wave (the reflection) with the same amplitude traveling to the left
- When they hit, the amplitude at node 20 adds and you get an amplitude of +200

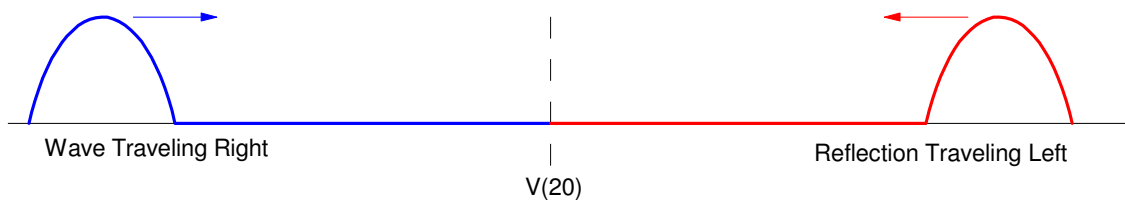


Figure 12: When $\Gamma = +1$, the transmission line behaves like an infinite transmission line with a reflection wave equal to +1 times the incident waveform traveling left

$\Gamma = -1$: When R_{20} is small (0.01 Ohm - meaning the point is stuck and cannot move)), the term Z_0 dominates and the reflection coefficient becomes -1

What this looks like is

- An infinite transmission line with a wave traveling to the right
- A second wave (the reflection) with the opposite amplitude traveling to the left
- When they hit, the amplitude at node 20 cancels and the point $V(20)$ remains at zero

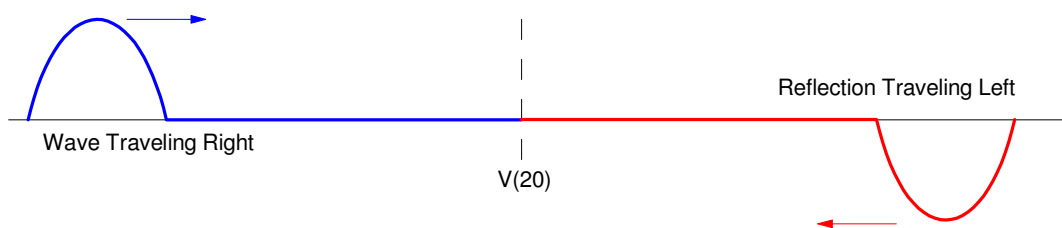


Figure 13: When $\Gamma = -1$, the transmission line behaves like an infinite transmission line with a reflection wave equal to -1 times the incident waveform traveling left

$\Gamma = 0$: To eliminate reflections, pick R_{20} so that there is no reflection. This is termed "impedance matching"

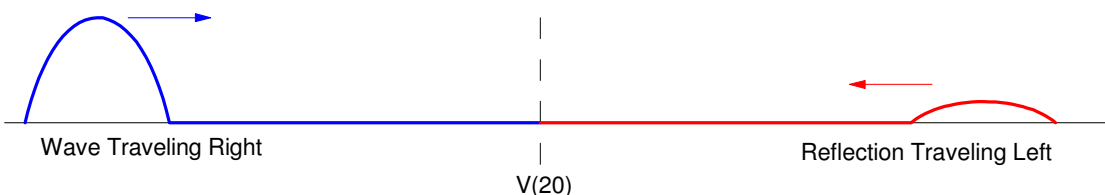


Figure 14: When $\Gamma = 0$, the transmission line behaves like an infinite transmission line with a reflection wave equal to zero times the incident waveform traveling left (i.e. there is no reflection)

Reflection Coefficient for $R_{30} = 0.1$:

Change R_{30} to be 0.1 (nearly zero), resulting in

$$\ddot{V}_{30} = \left(\frac{1}{LC}\right) V_{29} - \left(\frac{1}{LC}\right) V_{30} - \left(\frac{1}{RC}\right) \dot{V}_{30}$$

$$\ddot{V}_{30} = 50V_{29} - 100V_{30} - 100\dot{V}_{30}$$

```
V = zeros(30,1);
dV = zeros(30,1);
t = 0;
dt = 0.01;
while(t < 1000)
    for i=1:3

        if (t < 1.5) V0 = 100* ( sin(2*t).^2);
            else V0 = 0;
            end
        ddV(1) = 50*V0 - 100*V(1) + 50*V(2) - 0.01*dV(1);
        for i=2:29
            ddV(i) = 50*V(i-1) - 100*V(i) + 50*V(i+1) - 0.01*dV(i);
            end
        ddV(30) = 50*V(29) - 50*V(30) - 100*dV(30);
        for i=1:30
            dV(i) = dV(i) + ddV(i)*dt;
            V(i) = V(i) + dV(i)*dt;
            end
        t = t + dt;
    end

    hold on
    plot([0:30],[V0;V],'.-');
    ylim([-200,200]);
    pause(0.01);
end
```

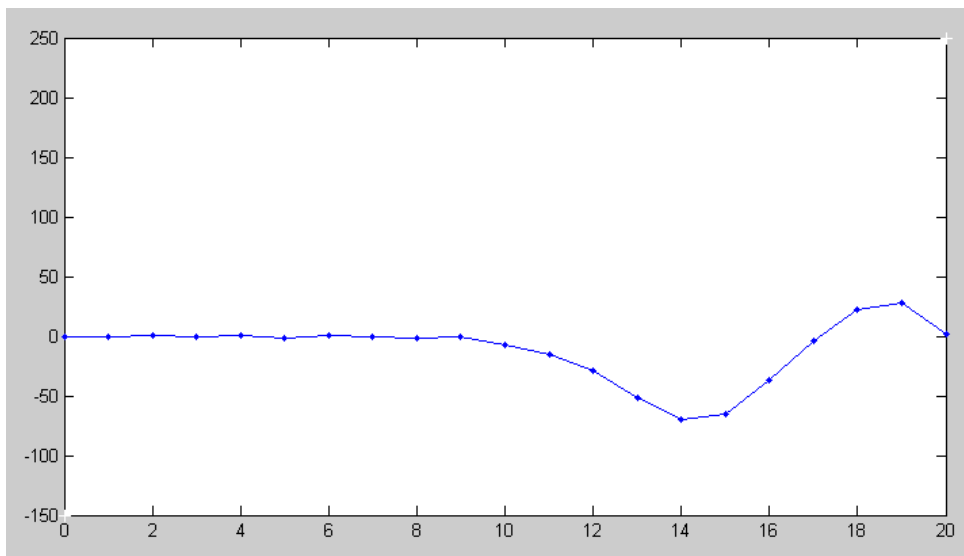


Figure 15: Voltage at $t = 4.0$ Seconds (after the wave reflects off the right end) with $R_{20} = 0.1$
Note that the reflection is $-100V$ (reflection = -1)

Somewhere between 1000 Ohms and 0.01 Ohm the reflection becomes zero. This is the ideal:

- If this is a clock line on a computer, you don't get false triggers
- If this is a transmission line on a power grid, you don't get over-voltages
- If this is a radio transmitter, you don't get reflected energy back to the transmitter

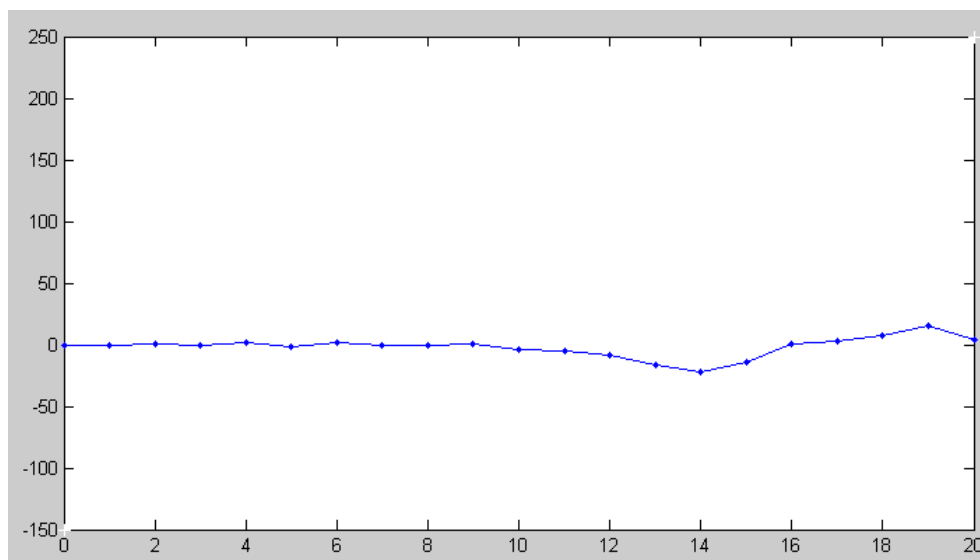


Figure 15: Voltage at $t = 4.0$ Seconds (after the wave reflects off the right end) with $R_{20} = 10$.
Note that the reflection is almost zero (reflection = 0)

Why Reflection Coefficients are Important

As mentioned previously, the wave equation describes a large number of systems including:

- Cars stuck in traffic
- Buildings swaying during an earthquake or a wind storm,
- Bridges oscillating during wind storms.
- Transmission lines galloping on windy days,
- Long electrical transmission lines, and
- Circuit traces

The reflections create standing waves, which result in

- Traffic coming to a standstill, clogging the highways
- Buildings swaying and collapsing if the swaying becomes excessive
- Bridges oscillating and then collapsing
- Transmission lines galloping, hitting each other or snapping due to metal fatigue,
- Large power surges on the nations high-voltage grid, and

In each one of these cases, the same trick can be used: if you can remove the discontinuity or absorb energy at the discontinuity, you can remove the reflections. If there are no reflections, there are no standing waves.

The reflections can also cause false triggers. For example, in ECE 376 you build your own microcontroller board (similar to a Raspberry Pi or an Arduino). If you max out the frequency of the board and look at the signals on an output pin, it looks like the following

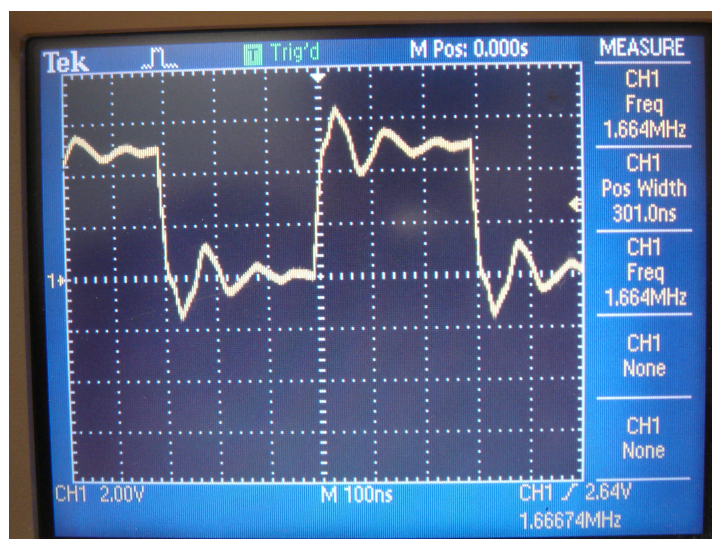


Figure 16: Voltage on an output pin on a PIC microcontroller used in ECE 376

The oscillations are due to traveling waves on the circuit board traces. If the voltages become too large, other devices will interpret this as multiple I/O signals - i.e. they'll fail to work.

This is major problem in industry:

- The clock frequency is constantly being increased.
- At some point, the circuit board traces start to look like waveguides rather than circuit traces. When that happens, the circuit that worked yesterday (at a lower clock frequency) no longer works today.

Dr. Braaten offers two courses in ECE that look specifically at this problem and how to mitigate it:

- ECE 453. Signal Integrity.
- ECE 455. Designing for Electromagnetic Compatibility.

These are two fairly unique courses taught at NDSU. These courses look at how to model circuits from the standpoint of electromagnetics (i.e. waveguides) and how to design them so they do not transmit radiation or have standing waves.

The ECE department also has its own shield room. This is essentially a large metal box:

- From physics, if you have a perfectly conducting sphere, there should be no electric fields inside the sphere.
- Any fields you *do measure* must come from inside the sphere.

With this room, students can build a circuit, close the door, and measure how strong the radiation is from that circuit. They can then use some tricks taught in ECE 453 and EEC 455 to redesign the circuit and see how much the radiation is reduced.

This also works with duality: if a circuit transmits is also receives. You can also hit the circuit (i.e. computer) with radiation and see how strong the fields have to be before the circuit stops working.



Figure 17: NDSU Shield Room in ECE