
Dynamics of a Double Pendulum

NDSU ECE 463/663

Lecture #10

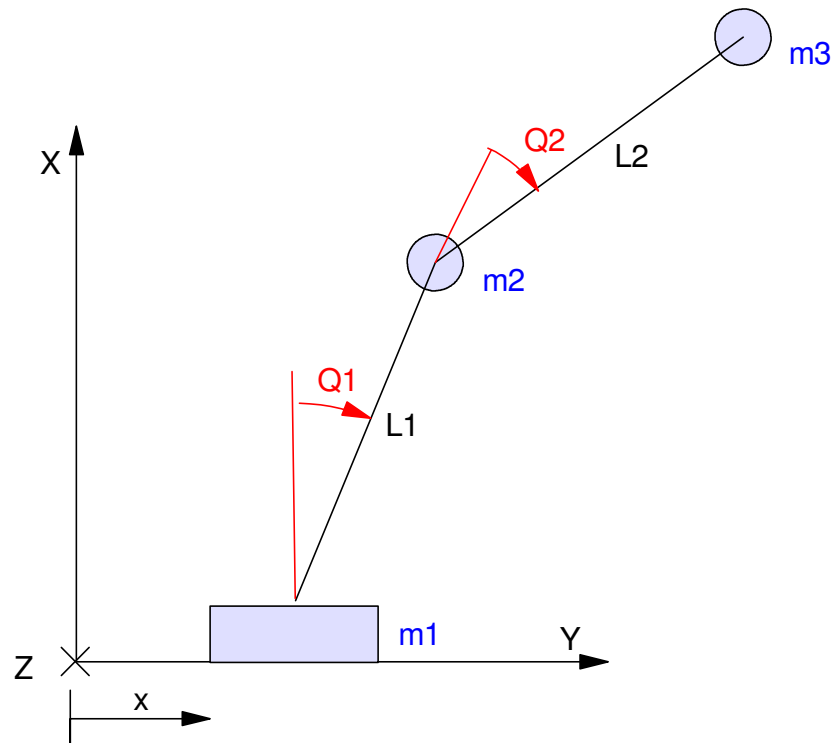
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Please visit [Bison Academy](#) for corresponding
lecture notes, homework sets, and solutions

Double Pendulum

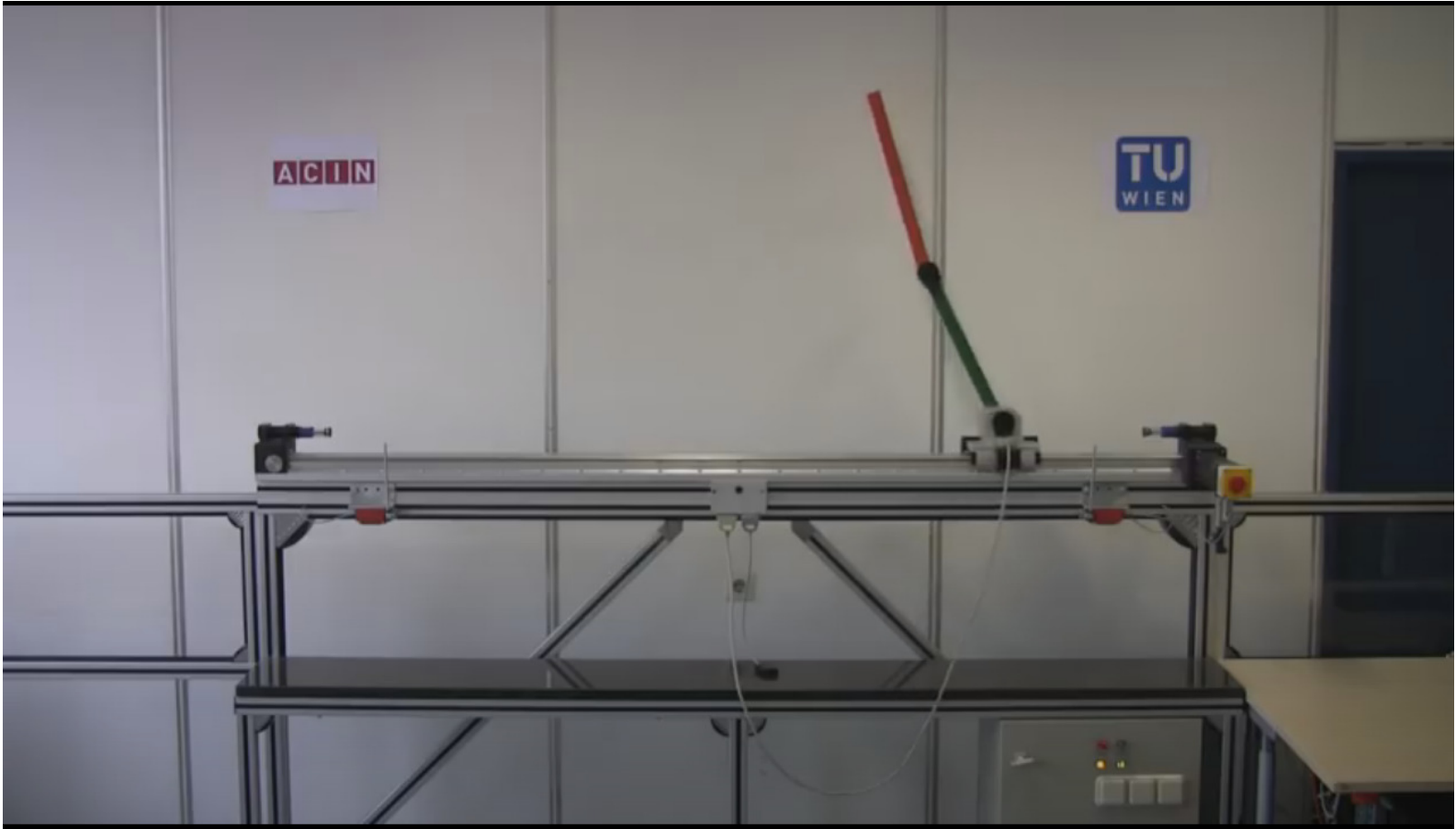
A Gantry system with a second mass added to it is shown below. Assume

- $m_1 = m_2 = m_3 = 1\text{kg}$
- $l_1 = l_2 = 1\text{m}$



Double Pendulum Example

<https://www.youtube.com/watch?v=B6vr1x6KDaY>



Derivation of Dynamics

Determine the position of each mass as a function of x , θ_1 , and θ_2

Mass 1:

$$y_1 = x$$

$$x_1 = 0$$

Mass 2:

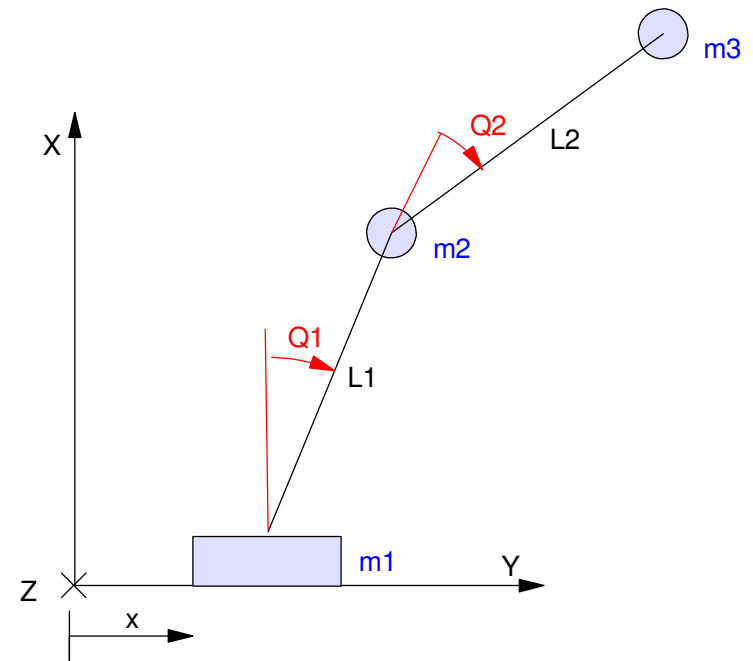
$$y_2 = y_1 + \sin(\theta_1)$$

$$x_2 = x_1 + \cos(\theta_1)$$

Mass 3:

$$y_3 = y_2 + \sin(\theta_1 + \theta_2)$$

$$x_3 = x_2 + \cos(\theta_1 + \theta_2)$$

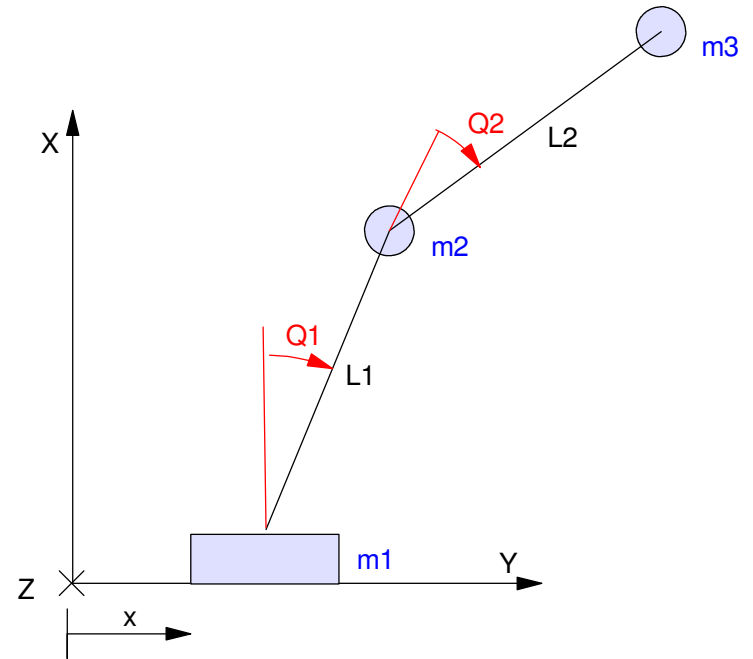


Simplifying and using shorthand notation

$$\begin{array}{ll} y_1 = x & x_1 = 0 \\ y_2 = x + s_1 & x_2 = c_1 \\ y_3 = x + s_1 + s_{12} & x_3 = c_1 + c_{12} \end{array}$$

Take the derivatives:

$$\begin{array}{ll} \dot{y}_1 = \dot{x} & \dot{x}_1 = 0 \\ \dot{y}_2 = \dot{x} + c_1 \dot{\theta}_1 & \dot{x}_2 = -s_1 \dot{\theta}_1 \\ \dot{y}_3 = \dot{x} + c_1 \dot{\theta}_1 + c_{12} (\dot{\theta}_1 + \dot{\theta}_2) & \\ \dot{x}_3 = -s_1 \dot{\theta}_1 - s_{12} (\dot{\theta}_1 + \dot{\theta}_2) & \end{array}$$



Kinetic Energy

$$KE = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2}m_3(\dot{x}_3^2 + \dot{y}_3^2)$$

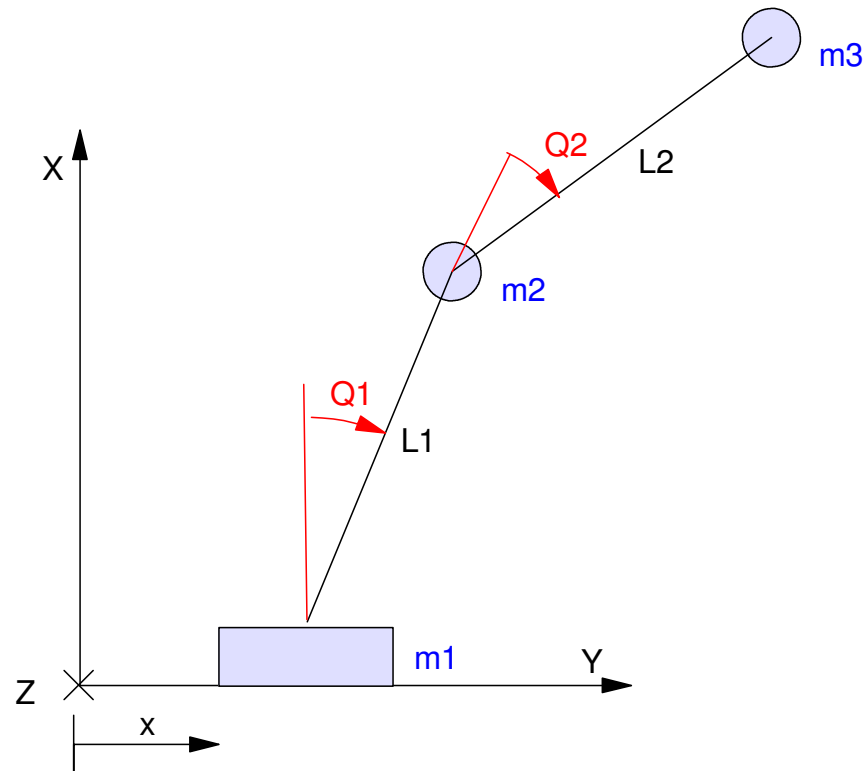
$$KE = \frac{1}{2}(\dot{x}^2) + \frac{1}{2}\left(\left(\dot{x} + c_1\dot{\theta}_1\right)^2 + \left(-s_1\dot{\theta}_1\right)^2\right) \\ + \frac{1}{2}\left(\left(\dot{x} + c_1\dot{\theta}_1 + c_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right)\right)^2 + \left(-s_1\dot{\theta}_1 - s_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right)\right)^2\right)$$

Simplifying

$$KE = \frac{3}{2}\dot{x}^2 + \dot{\theta}_1^2 + \frac{1}{2}\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + 2c_1\dot{x}\dot{\theta}_1 + c_2\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right) + c_{12}\dot{x}\left(\dot{\theta}_1 + \dot{\theta}_2\right)$$

Potential Energy

$$PE = m_1 g x_1 + m_2 g x_2 + m_3 g x_3$$



$$PE = g c$$

$$PE = 2g$$

Force on the Mass #1:

Form the LaGrangian:

$$L = \frac{3}{2}\dot{x}^2 + \dot{\theta}_1^2 + \frac{1}{2}(\dot{\theta}_1 + \dot{\theta}_2)^2 + 2c_1\dot{x}\dot{\theta}_1 + c_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2) + c_{12}\dot{x}(\dot{\theta}_1 + \dot{\theta}_2) - 2gc_1 - gc_{12}$$

Determine the force on the mass:

$$F = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x}$$

$$F = \frac{d}{dt}\left(3\dot{x} + 2c_1\dot{\theta}_1 + c_{12}(\dot{\theta}_1 + \dot{\theta}_2)\right) - 0$$

$$F = 3\ddot{x} + 2c_1\ddot{\theta}_1 + c_{12}(\ddot{\theta}_1 + \ddot{\theta}_2) - s_{12}(\dot{\theta}_1 + \dot{\theta}_2)^2$$

Torque on Q1

$$L = \frac{3}{2}\dot{\mathbf{x}}^2 + \dot{\theta}_1^2 + \frac{1}{2}(\dot{\theta}_1 + \dot{\theta}_2)^2 + 2c_1\dot{\mathbf{x}}\dot{\theta}_1 + c_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2) + c_{12}\dot{\mathbf{x}}(\dot{\theta}_1 + \dot{\theta}_2) - 2gc_1 - gc_{12}$$

$$T_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1}$$

$$T_1 = \frac{d}{dt}\left(2\dot{\theta}_1 + (\dot{\theta}_1 + \dot{\theta}_2) + 2c_1\dot{\mathbf{x}} + c_2(\dot{\theta}_1 + \dot{\theta}_2) + c_2\dot{\theta}_1 + c_{12}\dot{\mathbf{x}}\right) \\ - \left(-2s_1\dot{\mathbf{x}}\dot{\theta}_1 - s_{12}\dot{\mathbf{x}}(\dot{\theta}_1 + \dot{\theta}_2) + 2gs_1 + gs_{12}\right)$$

$$T_1 = 2\ddot{\theta}_1 + (\ddot{\theta}_1 + \ddot{\theta}_2) - 2s_1\dot{\mathbf{x}}\dot{\theta}_1 + 2c_1\ddot{\mathbf{x}} - s_2(\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_2 \\ + c_2(\ddot{\theta}_1 + \ddot{\theta}_2) - s_2\dot{\theta}_1\dot{\theta}_2 + c_2\ddot{\theta}_1 - s_{12}\dot{\mathbf{x}}(\dot{\theta}_1 + \dot{\theta}_2) + c_{12}\ddot{\mathbf{x}} \\ + 2s_1\dot{\mathbf{x}}\dot{\theta}_1 + s_{12}\dot{\mathbf{x}}(\dot{\theta}_1 + \dot{\theta}_2) - 2gs_1 - gs_{12}$$

Torque on Q2

$$L = \frac{3}{2}\dot{\mathbf{x}}^2 + \dot{\theta}_1^2 + \frac{1}{2}(\dot{\theta}_1 + \dot{\theta}_2)^2 + 2c_1\dot{\mathbf{x}}\dot{\theta}_1 + c_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2) + c_{12}\dot{\mathbf{x}}(\dot{\theta}_1 + \dot{\theta}_2) - 2gc_1 - gc_{12}$$

$$T_2 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2}$$

$$T_2 = \frac{d}{dt}\left((\dot{\theta}_1 + \dot{\theta}_2) + c_2\dot{\theta}_1 + c_{12}\dot{\mathbf{x}}\right) - \left(-s_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2) - s_{12}\dot{\mathbf{x}}(\dot{\theta}_1 + \dot{\theta}_2) + gs_{12}\right)$$

$$T_2 = (\ddot{\theta}_1 + \ddot{\theta}_2) - s_2\dot{\theta}_1\dot{\theta}_2 + c_2\ddot{\theta}_1 - s_{12}\dot{\mathbf{x}}(\dot{\theta}_1 + \dot{\theta}_2) + c_{12}\ddot{\mathbf{x}} \\ + s_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2) + s_{12}\dot{\mathbf{x}}(\dot{\theta}_1 + \dot{\theta}_2) - gs_{12}$$

Nonlinear Dynamics of a Double Pendulum:

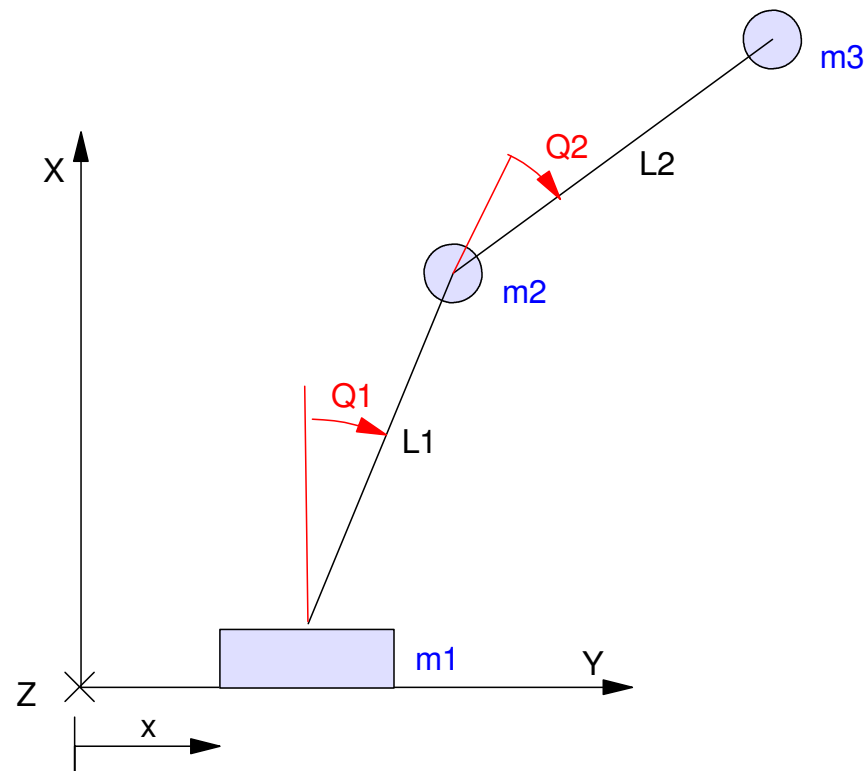
$$F = 3\ddot{x} + 2c_1\ddot{\theta}_1 - 2s_1\dot{\theta}_1^2 + c_{12}(\ddot{\theta}_1 + \ddot{\theta}_2) - s_{12}(\dot{\theta}_1 + \dot{\theta}_2)^2$$

$$\begin{aligned} T_1 = & 2\ddot{\theta}_1 + (\ddot{\theta}_1 + \ddot{\theta}_2) - 2s_1\dot{x}\dot{\theta}_1 + 2c_1\ddot{x} - s_2(\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_2 \\ & + c_2(\ddot{\theta}_1 + \ddot{\theta}_2) - s_2\dot{\theta}_1\dot{\theta}_2 + c_2\ddot{\theta}_1 - s_{12}\dot{x}(\dot{\theta}_1 + \dot{\theta}_2) + c_{12}\ddot{x} \\ & + 2s_1\dot{x}\dot{\theta}_1 + s_{12}\dot{x}(\dot{\theta}_1 + \dot{\theta}_2) - 2gs_1 - gs_{12} \end{aligned}$$

$$\begin{aligned} T_2 = & (\ddot{\theta}_1 + \ddot{\theta}_2) - s_2\dot{\theta}_1\dot{\theta}_2 + c_2\ddot{\theta}_1 - s_{12}\dot{x}(\dot{\theta}_1 + \dot{\theta}_2) + c_{12}\ddot{x} \\ & + s_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2) + s_{12}\dot{x}(\dot{\theta}_1 + \dot{\theta}_2) - gs_{12} \end{aligned}$$

Note:

- It's a rather nasty set of nonlinear dynamics
- LaGrangian dynamics were able to obtain the answer
- Other methods (free body diagrams, etc.) won't get you there



Robotics Notation for Dynamics

Another way to write this is

$$M\ddot{\Theta} = C(\theta, \dot{\theta}) + G(\theta) + T$$

where

- M is the mass (inertia) matrix
- $C()$ are the coriolis forces
- $G()$ is the gravity matrix, and
- T is the input (torques and forces).

Placing the dynamics in this form:

$$3\ddot{x} + 2c_1\ddot{\theta}_1 + c_{12}(\ddot{\theta}_1 + \ddot{\theta}_2) = F + 2s_1\dot{\theta}_1^2 + -s_{12}(\dot{\theta}_1 + \dot{\theta}_2)^2$$

$$\begin{aligned} 2\ddot{\theta}_1 + (\ddot{\theta}_1 + \ddot{\theta}_2) + 2c_1\ddot{x} + c_2(\ddot{\theta}_1 + \ddot{\theta}_2) + c_2\ddot{\theta}_1 + c_{12}\ddot{x} = & T_1 + 2s_1\dot{x}\dot{\theta}_1 + s_2(\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_2 \\ & + s_2\dot{\theta}_1\dot{\theta}_2 + s_{12}\dot{x}(\dot{\theta}_1 + \dot{\theta}_2) \\ & - 2s_1\dot{x}\dot{\theta}_1 - s_{12}\dot{x}(\dot{\theta}_1 + \dot{\theta}_2) + 2gs_1 + gs_{12} \end{aligned}$$

$$\begin{aligned} (\ddot{\theta}_1 + \ddot{\theta}_2) + c_2\ddot{\theta}_1 + c_{12}\ddot{x} = & T_2 + s_2\dot{\theta}_1\dot{\theta}_2 + s_{12}\dot{x}(\dot{\theta}_1 + \dot{\theta}_2) \\ & - s_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2) - s_{12}\dot{x}(\dot{\theta}_1 + \dot{\theta}_2) + gs_{12} \end{aligned}$$

Placing in matrix form:

$$M\ddot{\Theta} = C(\theta, \dot{\theta}) + G(\theta) + T$$

$$\begin{bmatrix} 3 & (2c_1 + c_{12}) & c_{12} \\ (2c_1 + c_{12}) & (3 + 2c_2) & (1 + c_2) \\ c_{12} & (1 + c_2) & 1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} =$$

$$\begin{bmatrix} +2s_1\dot{\theta}_1^2 + -s_{12}(\dot{\theta}_1 + \dot{\theta}_2)^2 \\ +2s_1\dot{x}\dot{\theta}_1 + s_2(\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_2 + s_2\dot{\theta}_1\dot{\theta}_2 - 2s_1\dot{x}\dot{\theta}_1 \\ +s_2\dot{\theta}_1\dot{\theta}_2 - s_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2) \end{bmatrix} + g \begin{bmatrix} 0 \\ +2s_1 + s_{12} \\ s_{12} \end{bmatrix} + \begin{bmatrix} F \\ T_1 \\ T_2 \end{bmatrix}$$

Matlab Code

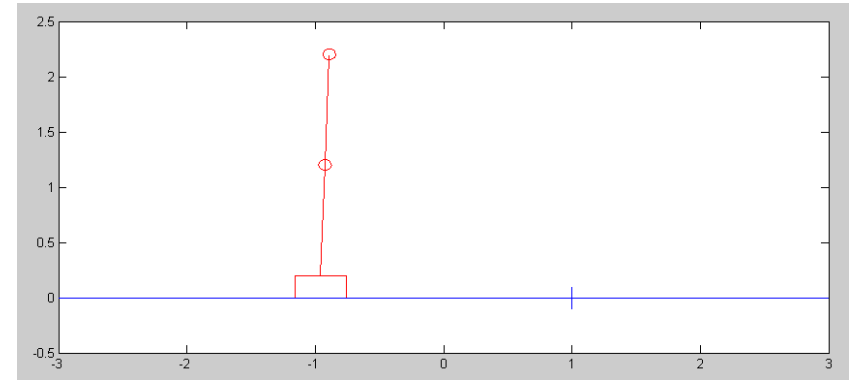
- Cart2 / Cart2Display / Cart2Dynamics
- Zero position points up

```
M = [3,          2*c1+c12,   c12;  
     2*c1+c12,   3+2*c2,    1+c2;  
     c12,        1+c2,      1];
```

```
C = [2*s1*dq1*dq1 - s12*(dq1+dq2)^2;  
     2*s1*dx*dq1 + s2*(dq1+dq2)*dq2 + s2*dq1*dq2 - 2*s1*dx*dq1;  
     s2*dq1*dq2 - s2*dq1*(dq1+dq2)];
```

```
G = [0 ; 2*s1 + s12 ; s12];
```

```
ddX = inv(M)*(C - g*G + [F ; 0; 0] );
```



Matlab Code

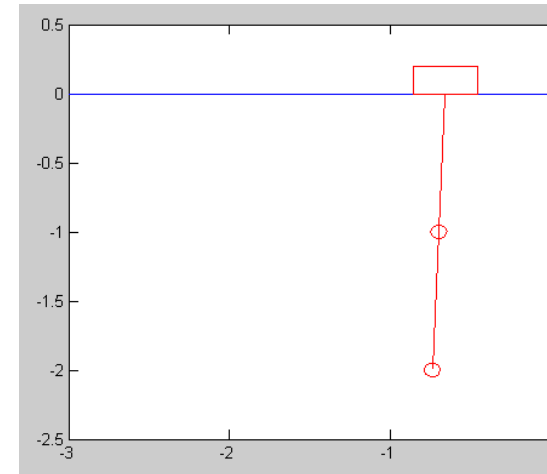
- Gantry2 / Gantry2Display / Gantry2Dynamics
- Same as Cart2 but change the direction of gravity
- Zero position points down as well

```
M = [3,          2*c1+c12,   c12;  
     2*c1+c12,   3+2*c2,    1+c2;  
     c12,        1+c2,      1];
```

```
C = [2*s1*dq1*dq1 - s12*(dq1+dq2)^2;  
     2*s1*dx*dq1 + s2*(dq1+dq2)*dq2 + s2*dq1*dq2 - 2*s1*dx*dq1;  
     s2*dq1*dq2 - s2*dq1*(dq1+dq2)];
```

```
G = [0 ; 2*s1 + s12 ; s12];
```

```
ddX = inv(M)*(C + g*G + [F ; 0; 0] );
```



Linearized Dynamics

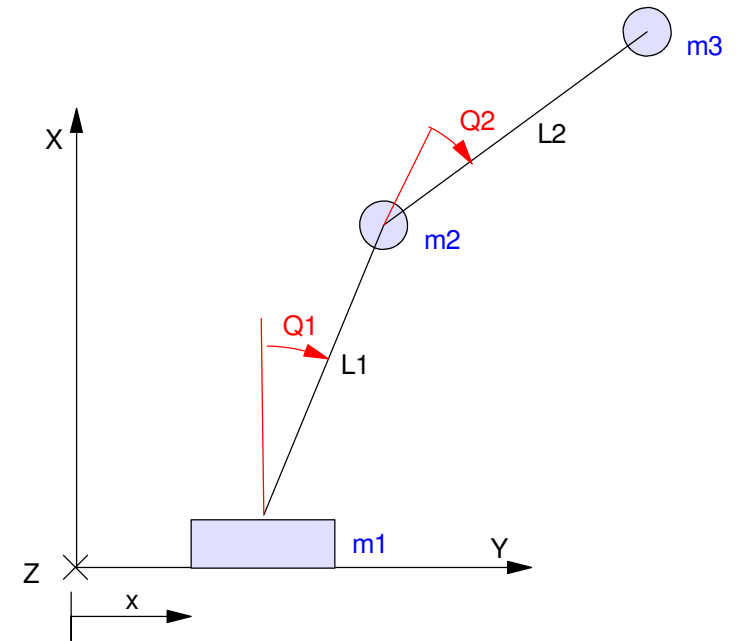
For small angles

$$c_1 \approx 1$$
$$\dot{\theta}_1 \dot{\theta}_2 \approx 0$$

$$s_1 \approx \theta_1$$

$$\begin{bmatrix} 3 & 3 & 1 \\ 3 & 5 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = g \begin{bmatrix} 0 \\ +2\theta_1 + \theta_1 + \theta_2 \\ \theta_1 + \theta_2 \end{bmatrix} + \begin{bmatrix} F \\ T_1 \\ T_2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 & 1 \\ 3 & 5 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = g \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ \theta_1 \\ \theta_2 \end{bmatrix} + \begin{bmatrix} F \\ T_1 \\ T_2 \end{bmatrix}$$



If the only input is the force on the base

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = g \begin{bmatrix} 0 & -2 & 0 \\ 0 & 3 & -1 \\ 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} x \\ \theta_1 \\ \theta_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} F$$

In State-Space form

$$s \begin{bmatrix} x \\ \theta_1 \\ \theta_2 \\ \dot{x} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2g & 0 & 0 & 0 & 0 \\ 0 & 3g & -g & 0 & 0 & 0 \\ 0 & -3g & 3g & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta_1 \\ \theta_2 \\ \dot{x} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} F$$

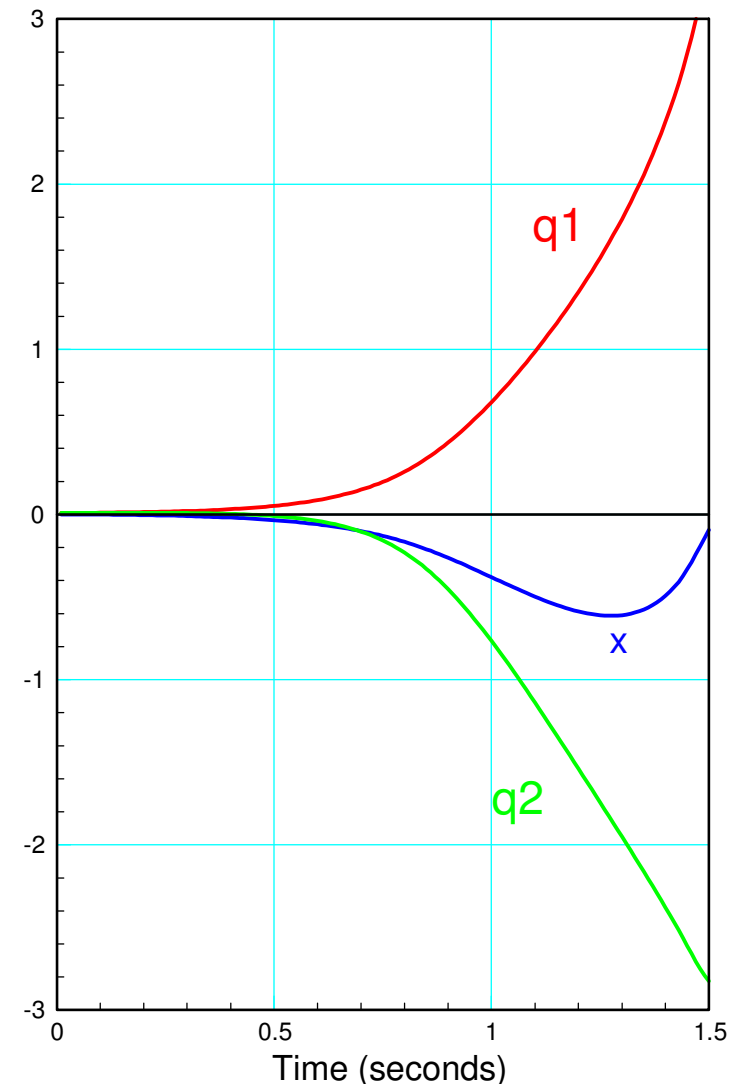
Checking the eigenvalues:

- Cart2 (unstable)

```
>> eig(A)
```

```
    0  
    0  
 -6.8099  
 -3.5250  
  6.8099  
  3.5250
```

The open-loop system is unstable as it should be. If you turn it off, it will fall.



If you reverse gravity, this is a double gantry system

```
>> A = [Z,I ; -inv(M)*G*9.8,Z]
```

```
      0      0      0      1.0000      0      0
      0      0      0      0      1.0000      0
      0      0      0      0      0      1.0000
      0     19.6000     0.0000      0      0      0
      0    -29.4000     9.8000      0      0      0
      0     29.4000    -29.4000      0      0      0
```

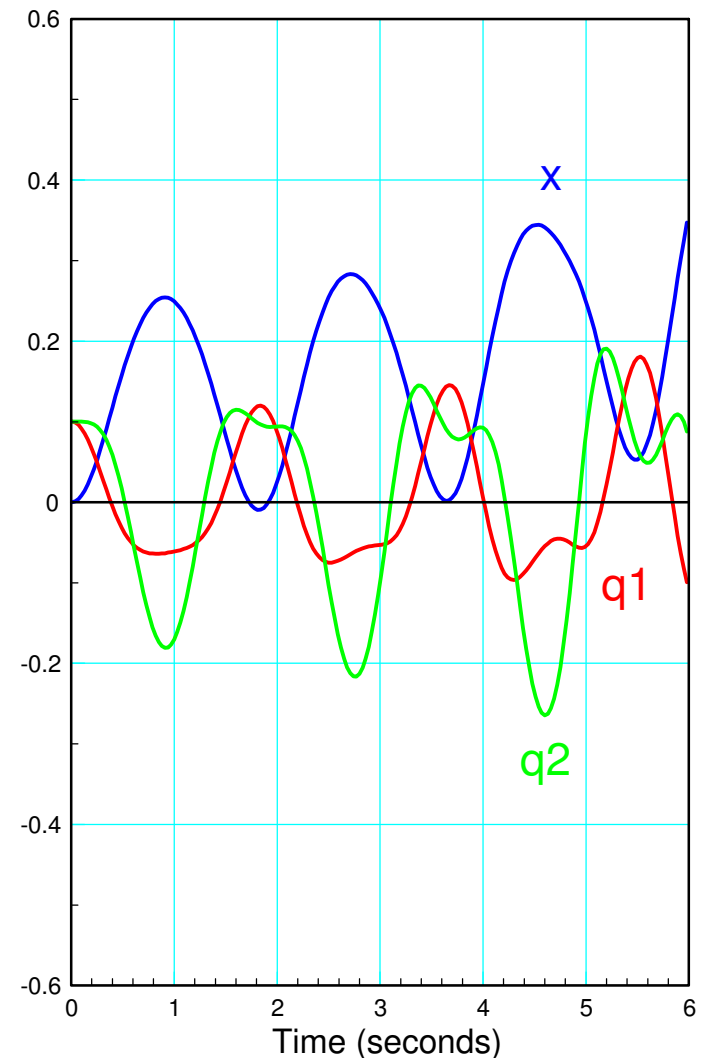
```
>> B = [0;0;0;1;-1;1]
```

```
0
0
0
1
-1
1
```

A double gantry system has complex poles

```
>> eig(A)
```

```
0  
0  
0.0000 + 6.8099i  
0.0000 - 6.8099i  
-0.0000 + 3.5250i  
-0.0000 - 3.5250i
```



Summary

- LaGrangian Formulation of Dynamics works again
- The derivation is a bit nasty, but it works
- That's one of the reasons engineers take so much math

