
Cart and Pendulum & Gantry Dynamics

NDSU ECE 463/663

Lecture #7

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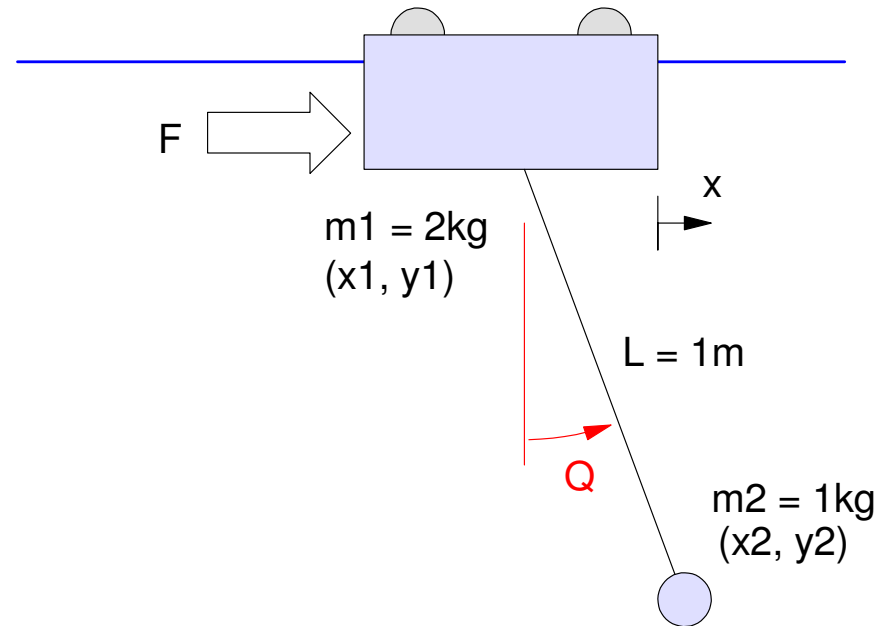
Please visit [Bison Academy](#) for corresponding
lecture notes, homework sets, and solutions

Gantry System

- Overhead lift found in factories
- Pick up an engine block
- Move it somewhere else
- Tends to oscillate

Problem: Find the dynamics

- $m_1 = 2\text{kg}$
- $m_2 = 1\text{kg}$
- $L = 1\text{m}$



LaGrangian Formulation

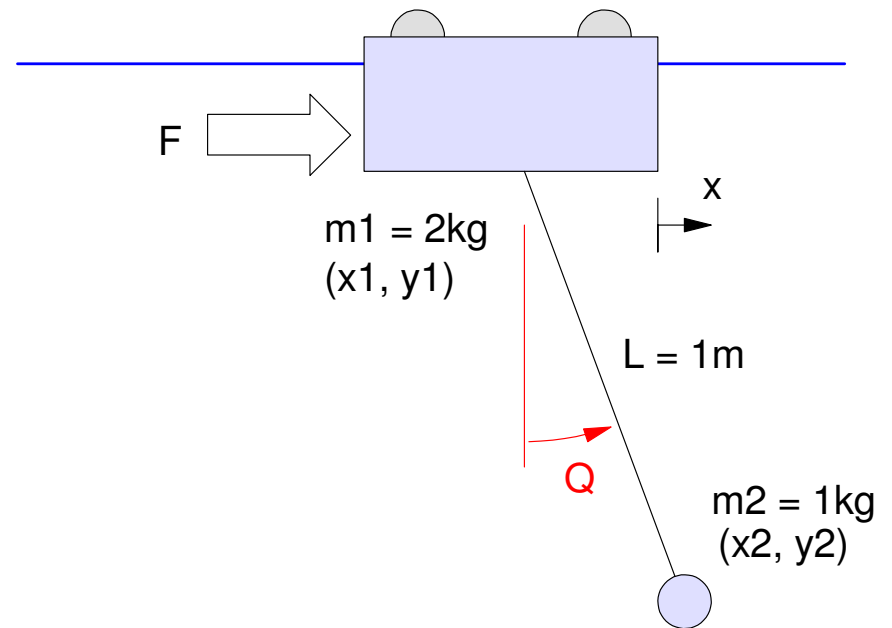
1) Define the energy in the system

Mass #1 (cart):

- position = (x_1, y_1)
- $x_1 = x$
- $y_1 = 0$

$$KE_1 = \frac{1}{2}mv^2 = \frac{1}{2} \cdot 2 \cdot \dot{x}^2$$

$$PE_1 = mgh = 0$$



Mass #2 (pendulum): position = (x_2, y_2)

- $x_2 = x + \sin \theta$ $\dot{x}_2 = \dot{x} + (\cos \theta)\dot{\theta}$
- $y_2 = -\cos \theta$ $\dot{y}_2 = (\sin \theta)\dot{\theta}$

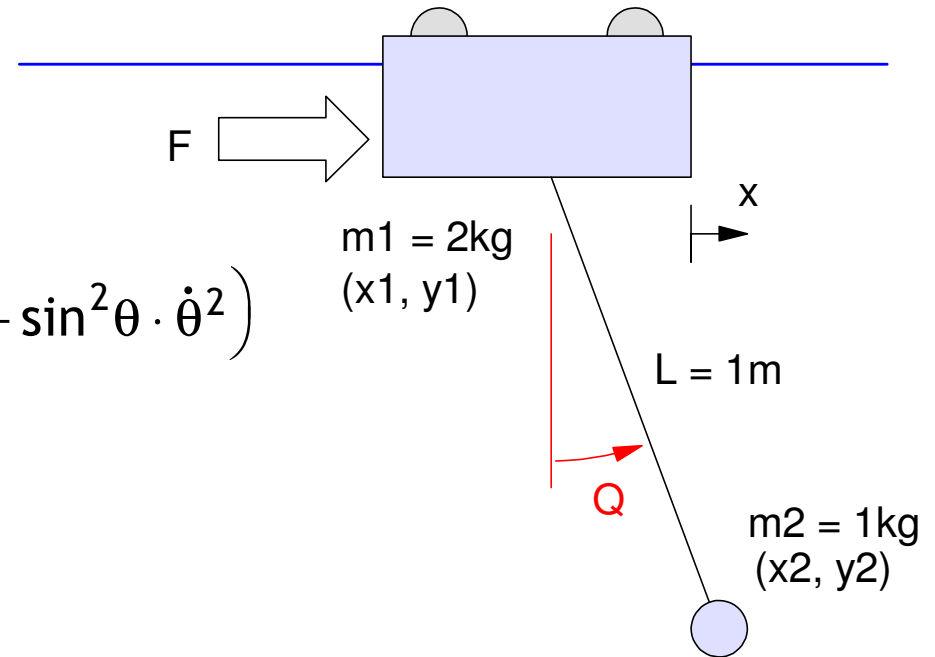
$$KE_2 = \frac{1}{2}(\dot{x}_2^2 + \dot{y}_2^2)$$

$$KE_2 = \frac{1}{2}(\dot{x}^2 + 2\dot{x}\dot{\theta}\cos\theta + \cos^2\theta \cdot \dot{\theta}^2 + \sin^2\theta \cdot \dot{\theta}^2)$$

$$KE_2 = \frac{1}{2}(\dot{x}^2 + 2\dot{x}\dot{\theta}\cos\theta + \dot{\theta}^2)$$

$$PE_2 = mgh = gy_2$$

$$PE_2 = -g\cos\theta$$



So, the Lagrangian is

$$L = \left(\dot{x}^2 + \frac{1}{2} \left(\dot{x}^2 + 2\dot{x}\dot{\theta} \cos \theta + \dot{\theta}^2 \right) \right) - (-g \cos \theta)$$

$$L = \left(\frac{3}{2} \dot{x}^2 + \dot{x}\dot{\theta} \cos \theta + \frac{1}{2} \dot{\theta}^2 \right) + g \cos \theta$$

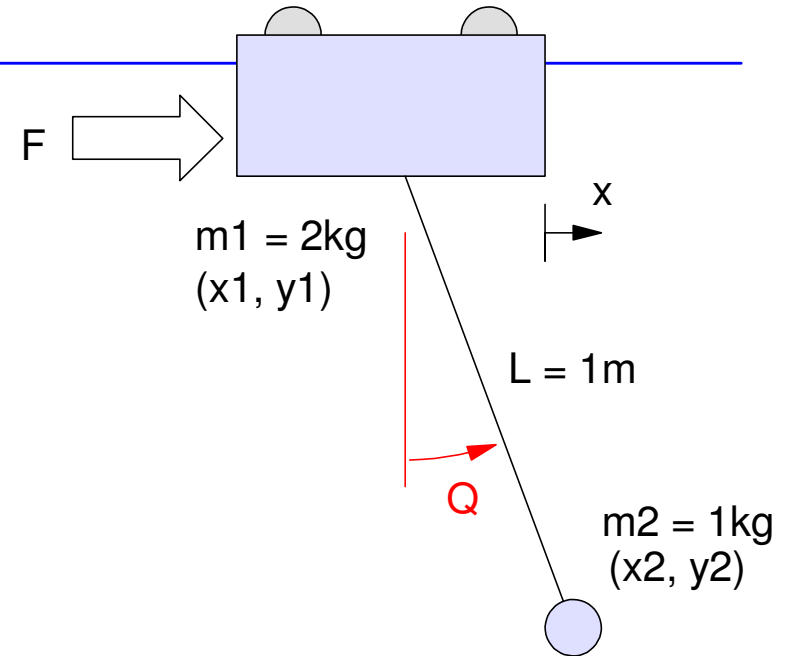
The force on the base is

$$L = \left(\frac{3}{2} \dot{x}^2 + \dot{x}\dot{\theta} \cos \theta + \frac{1}{2} \dot{\theta}^2 \right) + g \cos \theta$$

$$F = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \left(\frac{\partial L}{\partial x} \right)$$

$$F = \frac{d}{dt} \left(3\dot{x} + \dot{\theta} \cos \theta \right) - 0$$

$$F = 3\ddot{x} + \ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta$$



The torque on the beam is

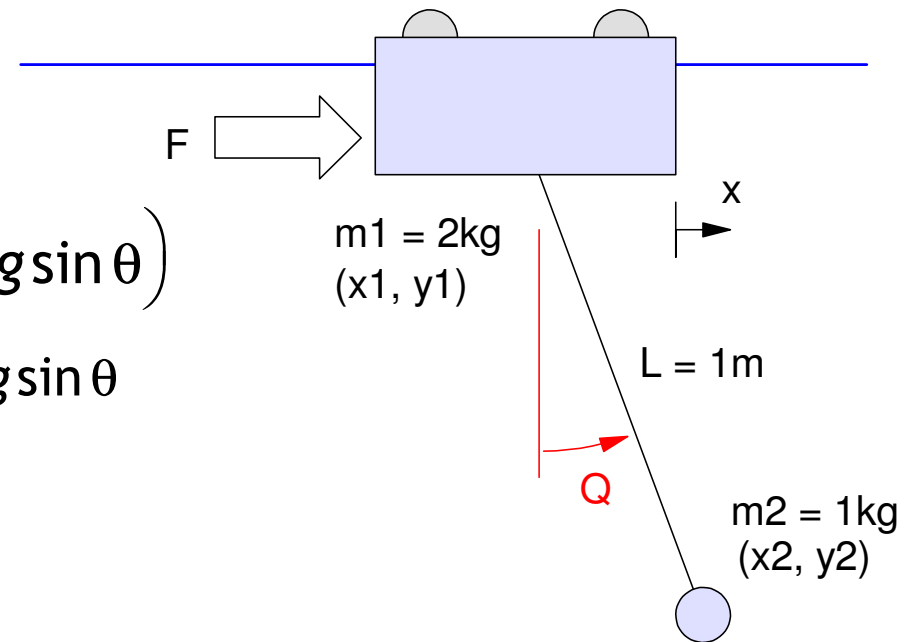
$$L = \left(\frac{3}{2} \dot{x}^2 + \dot{x} \dot{\theta} \cos \theta + \frac{1}{2} \dot{\theta}^2 \right) + g \cos \theta$$

$$T = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \left(\frac{\partial L}{\partial \theta} \right)$$

$$T = \frac{d}{dt} \left(\dot{x} \cos \theta + \dot{\theta} \right) - \left(-\dot{x} \dot{\theta} \sin \theta - g \sin \theta \right)$$

$$T = \ddot{x} \cos \theta - \dot{x} \dot{\theta} \sin \theta + \ddot{\theta} + \dot{x} \dot{\theta} \sin \theta + g \sin \theta$$

$$T = \ddot{x} \cos \theta + \ddot{\theta} + g \sin \theta$$



So, the dynamics are

- This is what you use to simulate the system

$$\begin{bmatrix} 3 & \cos \theta \\ \cos \theta & 1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta}^2 \sin \theta \\ -g \sin \theta \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F$$

Linear Model

For small perturbations about 0,

$$\sin \theta \approx \theta$$

$$\cos \theta \approx 1$$

$$\dot{\theta}^2 \approx 0$$

$$g = 9.8 \text{ m/s}^2$$

The dynamics linearized about zero are then

$$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ -9.8\theta \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} 0 \\ 9.8\theta \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F \right)$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 4.9\theta + 0.5F \\ -14.7\theta - 0.5F \end{bmatrix}$$

In State-Space form

$$s \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 4.9 & 0 & 0 \\ 0 & -14.7 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ -0.5 \end{bmatrix} F$$

Note that the eigenvalues of 'A' are

$$\{0, 0, j3.83, -j3.83\}$$

The pendulum swings back and forth at 3.83 rad/sec.

Eigenvalues and Eigenvectors

$$\lambda = \{0, 0, j3.83, -j3.38\}$$

$$\Lambda = \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.316 \\ -0.950 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0.316 \\ -0.950 \end{pmatrix} \right\}$$

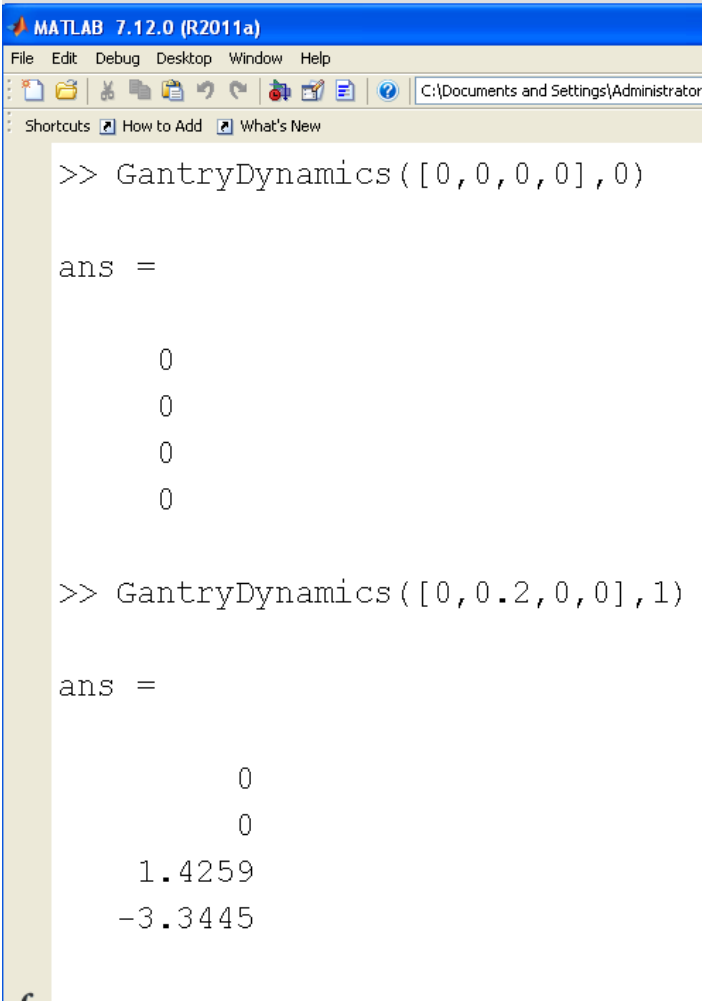
This means

- If you displace the cart by +1m, that state decays as e^{0t} (first eigenvector)
 - If you start the cart moving right at 1m/s, it keeps drifting right decaying as e^{0t} (2nd eigenvector)
 - If you displace the cart 0.316m right and swing the angle 0.950 rad left, the cart oscillates back and forth at 3.83 rad/sec (3rd eigenvector)
 - If you start out at (0, 0) but make the initial velocity 0.316 m/s right and the initial angular velocity -0.950 rad/sec, the cart oscillates back and forth at 3.83 rad/sec (4th eigenvector)
-

Nonlinear Dynamics with Matlab

$$\begin{bmatrix} 3 & \cos \theta \\ \cos \theta & 1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta}^2 \sin \theta \\ -g \sin \theta \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F$$

```
function [ dX ] = GantryDynamics( X, F )
%cart dynamics
% cart = 2kg
% ball = 1kg
% length = 1m
% X = [x, q, dx, dq]
x = X(1);
q = X(2);
dx = X(3);
dq = X(4);
g = 9.8;
M = [3, cos(q); cos(q), 1];
A = [dq*dq*sin(q); -g*sin(q)];
B = [1;0];
d2X = inv(M) * (A + B*F);
dX = [dx; dq; d2X];
end
```



The image shows a screenshot of the MATLAB 7.12.0 (R2011a) software interface. The window title is "MATLAB 7.12.0 (R2011a)". The menu bar includes "File", "Edit", "Debug", "Desktop", "Window", and "Help". The toolbar contains various icons for file operations and debugging. The command window shows the following commands and outputs:

```
>> GantryDynamics([0,0,0,0],0)

ans =

     0
     0
     0
     0

>> GantryDynamics([0,0.2,0,0],1)

ans =

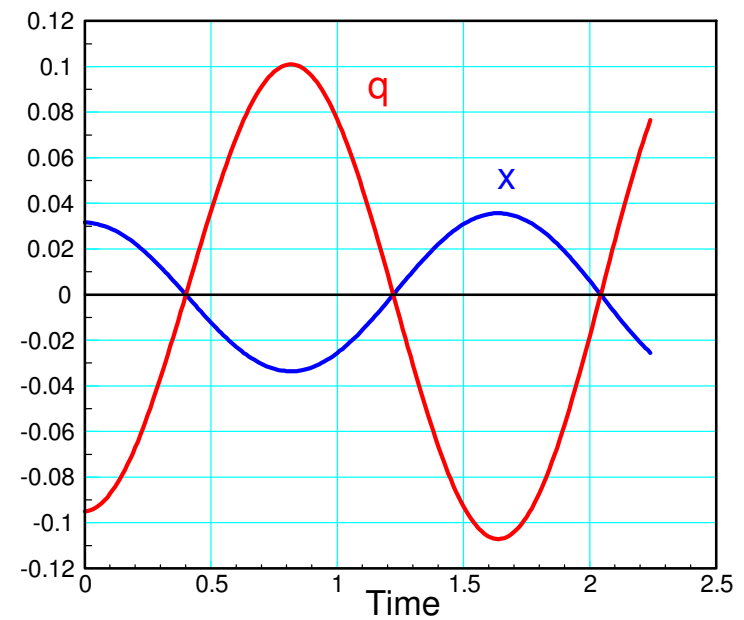
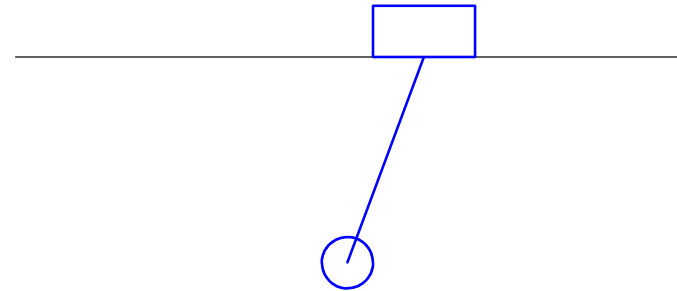
     0
     0
    1.4259
   -3.3445
```

Animation and Eigenvectors

```
X = [0.0316;-0.0950;0;0];
dX = zeros(4,1);
Ref = 0;
dt = 0.01;
U = 0;
t = 0;
y = [];

while(t < 10)
    dX = GantryDynamics(X, U);
    X = X + dX * dt;
    t = t + dt;
    GantryDisplay(X, Ref);
    y = [y ; X(1), X(2) ];
end

hold off
t = [0:length(y)-1]' * dt;
plot(t,y);
```



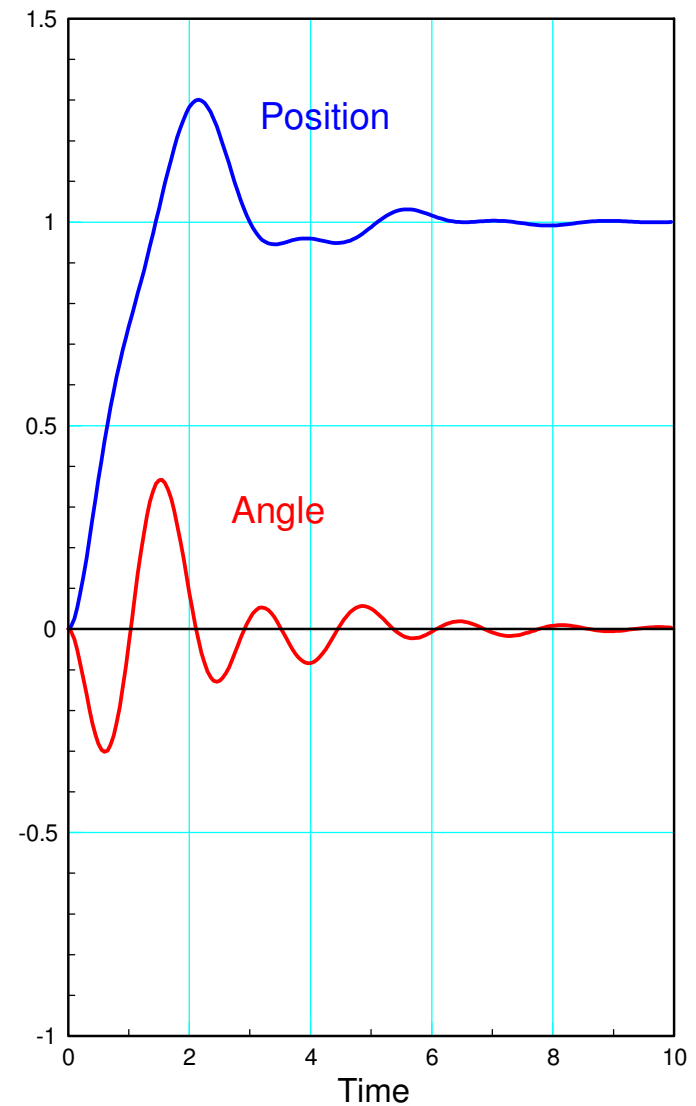
PD Control of a Gantry System

$$F = P(R - x) + D(\dot{R} - \dot{x})$$

Trial and error to find P and D

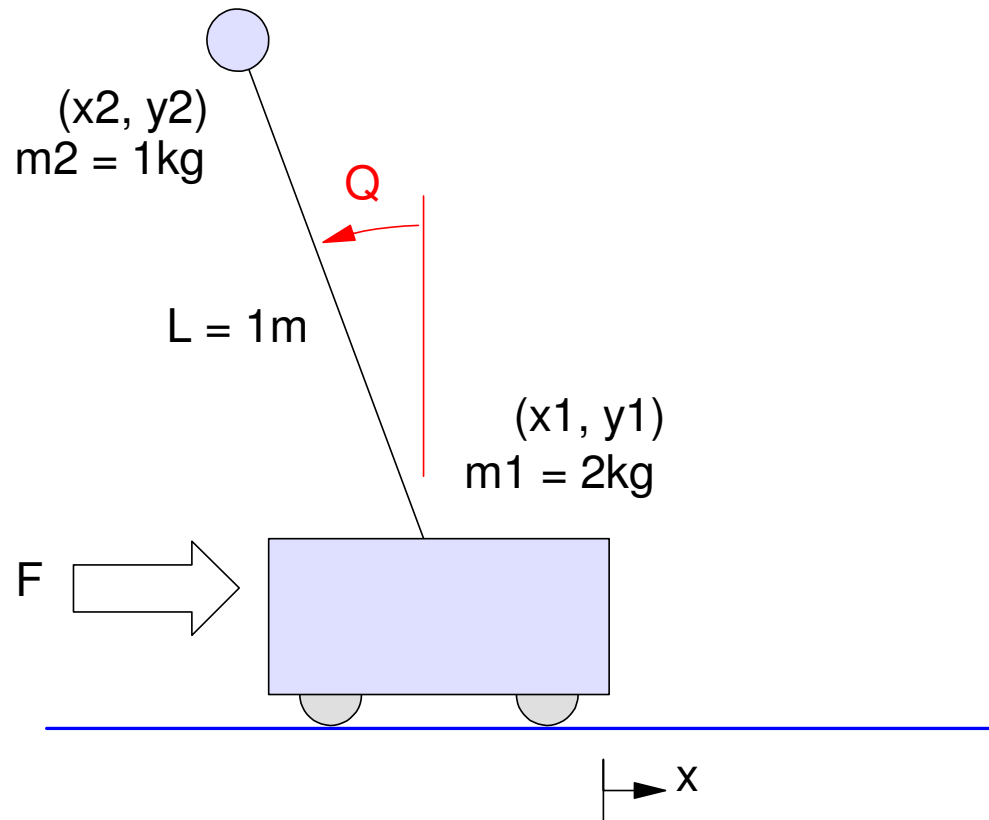
- ECE 461 gives other methods

```
X = [0;0;0;0];  
dX = zeros(4,1);  
Ref = 1;  
dt = 0.01;  
U = 0;  
t = 0;  
y = [];  
P = 10;  
D = 5;  
  
while(t < 10)  
    U = P*(Ref - X(1)) + D*(0 - X(3));  
    dX = GantryDynamics(X, U);  
    X = X + dX * dt;  
    t = t + dt;  
    GantryDisplay(X, Ref);  
    y = [y ; X(1), X(2) ];  
end
```



Cart & Pendulum (Cart.m)

- Balance a rocket as you move it to the launch pad
- Balance a yard stick on your hand



Same dynamics as before, just change the direction of gravity

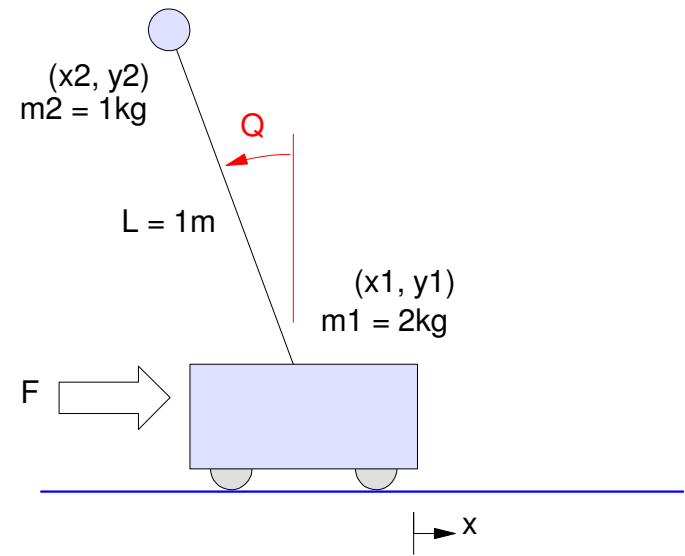
$g = -9.8 \text{ m/s}^2$:

$$s \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -4.9 & 0 & 0 \\ 0 & 14.7 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ x' \\ \theta' \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ -0.5 \end{bmatrix} F$$

Eigenvalues

$\{0, 0, 3.83, -3.83\}$

As expected, an inverted pendulum is unstable.



Eigenvalues and Eigenvectors

$$\lambda = \{0, 0, +3.83, -3.38\}$$

$$\Lambda = \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -0.0798 \\ 0.2394 \\ -0.3060 \\ 0.9180 \end{pmatrix}, \begin{pmatrix} 0.0798 \\ -0.2394 \\ -0.3060 \\ 0.9180 \end{pmatrix} \right\} \text{ change}$$

This means

- In theory, you can get the pendulum to balance if you get the initial condition just right (the 4th eigenvector)
- In practice, any non-zero initial condition on the 3rd eigenvector and the pendulum falls over

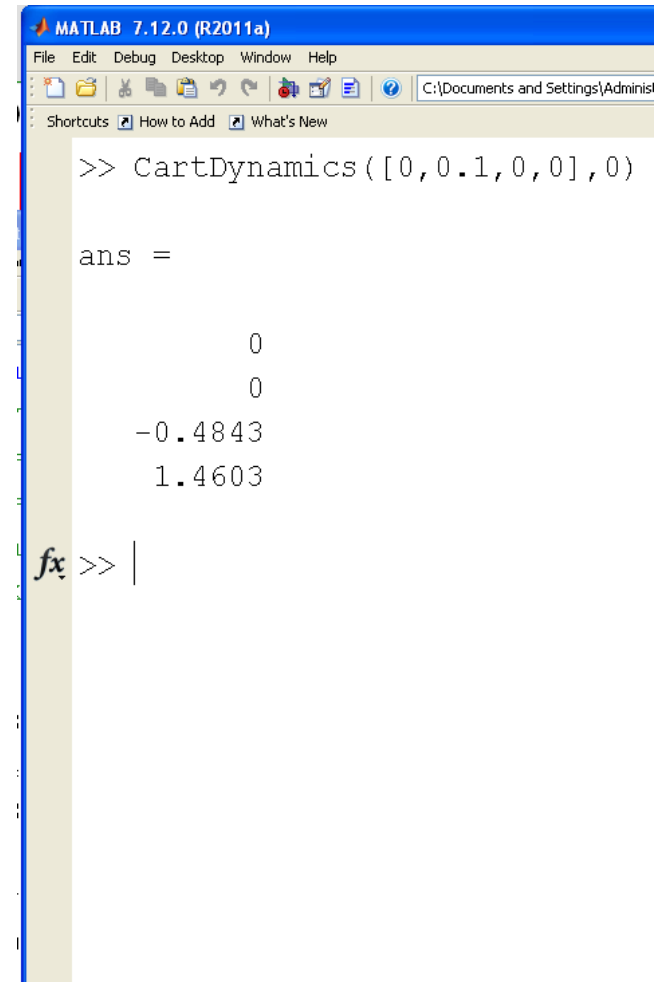
Nonlinear Dynamics in Matlab

$$\begin{bmatrix} 3 & \cos \theta \\ \cos \theta & 1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta}^2 \sin \theta \\ +g \sin \theta \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F$$

Only change: Sign of gravity term

- Results in positive feedback (unstable)

```
function [ dX ] = CartDynamics( X, F )
%cart dynamics
% cart = 2kg
% ball = 1kg
% length = 1m
% X = [x, q, dx, dq]
x = X(1);
q = X(2);
dx = X(3);
dq = X(4);
g = 9.8;
M = [3, cos(q); cos(q), 1];
A = [dq*dq*sin(q); g*sin(q)];
B = [1;0];
d2X = inv(M) * (A + B*F);
dX = [dx; dq; d2X];
end
```

A screenshot of the MATLAB 7.12.0 (R2011a) command window. The window title is "MATLAB 7.12.0 (R2011a)". The menu bar includes "File", "Edit", "Debug", "Desktop", "Window", and "Help". The toolbar shows various icons for file operations and debugging. The command window shows the following text: ">> CartDynamics([0,0.1,0,0],0)", "ans =", and a 4x1 column vector: 0, 0, -0.4843, 1.4603. Below this, the prompt "fx >>|" is visible.

```
MATLAB 7.12.0 (R2011a)
File Edit Debug Desktop Window Help
Shortcuts How to Add What's New

>> CartDynamics([0,0.1,0,0],0)

ans =

         0
         0
    -0.4843
     1.4603

fx >>|
```

Open-Loop Response

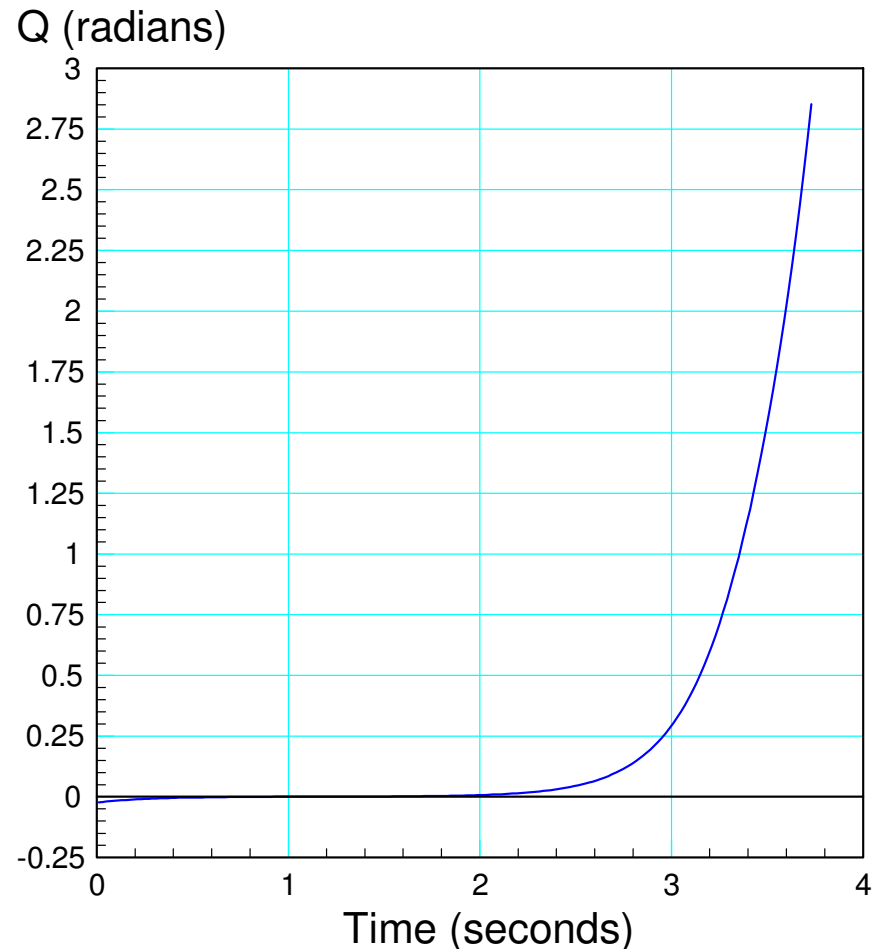
- $U = 0$

The system is unstable

Any initial condition results in the pendulum falling

PD Control also doesn't work

- $U = P(R - x) + D(\dot{R} - \dot{x})$
- No set of PD gains will stabilize this system



Full-State Feedback

It is possible to stabilize this system

- One solution uses all four states (full-state feedback)
- Upcoming topic for ECE 463

