Pole Placement (Bass Gura)

Definition:

Open-Loop System: System dynamics with U = 0.

$$sX = AX$$

Closed-Loop System: System dynamics with U = -Kx X

$$sX = (A - BK_x)X$$

Characteristic Polynomial:

a) The polynomial with roots equal to the eigenvalues of A

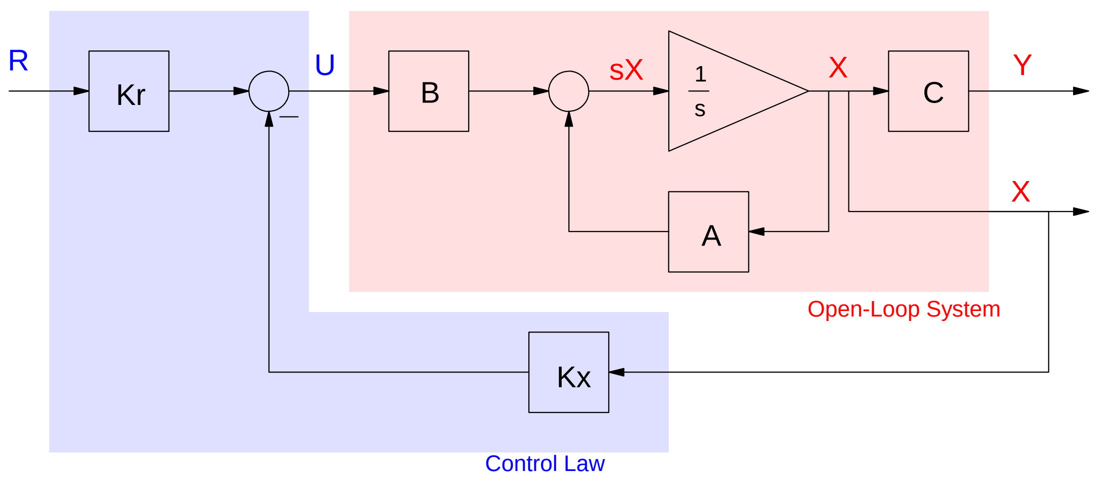
```
poly(eig(A))
```

b) The denominator polynomial of the transfer function

Bass Gura Derivation:

Assume a system is controllable. Can you place the closed-loop poles wherever you like using full-state feedback as well as set the DC gain from R to Y using the control law?

$$U = K_r R - K_x X$$



Problem: Find Kx and Kr to Place the Poles of the Closed-Loop System and Set the DC Gain from R to Y

Case 1: Controller Canonical Form:

Assume the system is in controller canonical form with a characteristic polynomial (i.e. the denominator of the transfer function) of

$$P(s) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

Find the feedback gains so that the characteristic polynomial is equal to a desired polynomial:

$$P_d(s) = s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0$$

The solution is fairly easy to see in state-space form. Since we assume the system is in controller canonical form, the plant dynamics are:

$$sX = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} U$$

With full-state feedback, this becomes

$$U = -\begin{bmatrix} k_0 & k_1 & k_2 & k_3 \end{bmatrix} X$$

or, substituting

$$sX = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 - k_0 & -a_1 - k_1 & -a_2 - k_2 & -a_3 - k_3 \end{bmatrix} X$$

The characteristic polynomial of the closed-loop system can be seen by observation to be:

$$s^4 + (a_3 + k_3)s^3 + (a_2 + k_2)s^2 + (a_1 + k_1)s + (a_0 + k_0) = 0$$

which is to be equal to the desired characteristic polynomial:

$$s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0 = 0$$

Matching terms results in the feedback gains being the difference between the desired and open-loop characteristic polynomials.

$$k_3 = b_3 - a_3$$

$$k_2 = b_2 - a_2$$

$$k_1 = b_1 - a_1$$

$$k_0 = b_0 - a_0$$

Case 2: The system is controllable but not in controller canonical form

If this is the case,

- First, find a similarity transform, T, which takes you to controller canonical form
- Second, find the feedback gains to place the closed-loop poles in controller form
- Finally, convert these feedback gains to state-variable form with this similarity transform.

This method is called Bass-Gura or Pole Placement.

Step 1: Find a similarity transform which takes you to controller canonical form. One which does this is

$$T = T_1 T_2 T_3$$

where T1 is the controllability matrix (assume a 4th-order system here):

$$T_1 = \left[\begin{array}{cccc} B & AB & A^2B & A^3B \end{array}\right]$$

T2 is related to the system's characteristic polynomial

$$T_2 = \begin{bmatrix} 1 & b_3 & b_2 & b_1 \\ 0 & 1 & b_3 & b_2 \\ 0 & 0 & 1 & b_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

T3 is a flip matrix

$$T_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

This similarity transform results in the transformed system being

$$Z = TX$$

$$sZ = (T^{-1}AT)Z + (T^{-1}B)U$$

or

$$sZ = A_z Z + B_z U$$

where (Az, Bz) are in controller canonical form.

Note that since T includes the controllability matrix and T is inverted, the (A, B) must be controllable for this algorithm to work.

Step 2: The full-state feedback gains in controller form is the difference between the current and desired characteristic polynomials

Open-Loop Characteristic Polynomial:

$$P(s) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

Closed-Loop (desired) Characteristic Polynomial:

$$P_d(s) = s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0$$

Feedback Gains:

$$K_z = \left[\ (b_0 - a_0) \ \ (b_1 - a_1) \ \ (b_2 - a_2) \ \ (b_3 - a_3) \ \right]$$

Step 3: Convert back to state-variable form (X) using the similarity transform:

$$K_x = K_z T^{-1}$$

Step 4: Check your answer. The closed-loop system is then

$$sX = (A - BK_x)X$$

The eigenvalues of $(A - BK)$ should be where you wanted to place them.

Step 5: Set the DC gain from R to Y to be equal to one. Add in the term Kr R to U

$$U = K_r R - K_x X$$

where R is the reference input (the set point). With this control law, the closed-loop system is

$$sX = (A - BK_x)X + BK_r R$$

$$Y = CX$$

The steady-state (i.e. DC) gain is

$$sX = 0 = (A - BK_x)X + BK_r R$$

Solving for X:

$$X = -(A - BK_x)^{-1} BK_r R$$

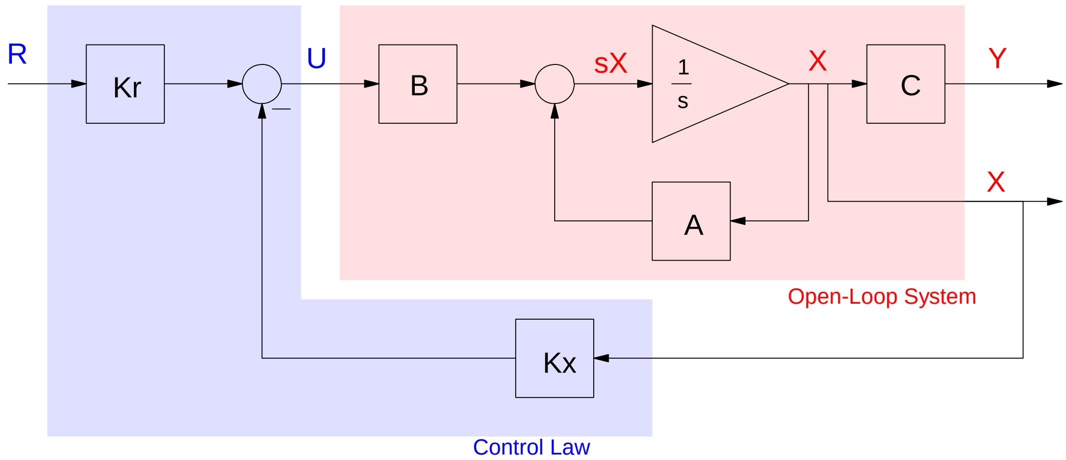
resulting in the output, Y, being

$$Y = -C(A - BK_x)^{-1} BK_r R$$

If the DC gain is to be one, then pick Kr so that

$$-C(A - BK_x)^{-1} BK_r = 1$$

The open-loop system plus the feedback control law then looks like the following



Feedback Control Law to Place the Poles of the Closed-Loop System (Kx) and Set the DC Gain (Kr)

Example 1: Heat Equation.

Assume a system has the following dynamics:

$$sX = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} U$$

Find the feedback gain, Kx, to place the poles of the closed-loop system

$$U = -K_x X$$

at { -1, -2, -3, -4}

Step 0: Input the system into Matlab

```
>> A = [-2,1,0,0; 1,-2,1,0; 0,1,-2,1; 0,0,1,-1]

    -2     1     0     0
     1    -2     1     0
     0     1    -2     1
     0     0     1    -1

>> B = [1;0;0;0]

     1
     0
     0
     0
```

Step 1: Find the similarity transform which takes you to controller canonical form.

T1 is the controllability matrix:

```
>> T1 = [B, A*B, A*A*B, A*A*A*B]

     1    -2     5   -14
     0     1    -4    14
     0     0     1    -6
     0     0     0     1
```

T2 is related to the system's characteristic polynomial

```
>> P = poly(eig(A))

    1.0000    7.0000   15.0000   10.0000    1.0000

>> T2 = [ P(1:4);  0, P(1:3);  0, 0, P(1:2);  0, 0, 0, P(1)]

    1.0000    7.0000   15.0000   10.0000
         0    1.0000    7.0000   15.0000
         0         0    1.0000    7.0000
         0         0         0    1.0000
```

T3 is a flip matrix

```
>> T3 = [0,0,0,1;0,0,1,0;0,1,0,0;1,0,0,0]

     0     0     0     1
     0     0     1     0
     0     1     0     0
     1     0     0     0
```

With T1, T2, T3, you can create the transform which takes you to controller canonical form:

```
>> T = T1*T2*T3;

>> Az = inv(T)*A*T

         0    1.0000         0         0
         0   -0.0000    1.0000    0.0000
         0    0.0000    0.0000    1.0000
   -1.0000  -10.0000  -15.0000   -7.0000

>> Bz = inv(T)*B;

         0
         0
         0
         1
```

Yup: (Az, Bz) are in controller canonical form.

Step 2: Find the full-state feedback gains in controller form. This is the difference between the desired and open-loop characteristic polynomials:

```
>> Pd = poly([-1, -2, -3, -4])

     1    10    35    50    24

>> P = poly(eig(A))

     1     7    15    10     1

>> dP = Pd - P

     0     3    20    40    23


>> Kz = dP([5,4,3,2])

    23    40    20     3
```

Check that Kz is correct:

```
>> eig(Az - Bz*Kz)

   -4.0000
   -3.0000
   -2.0000
   -1.0000
```

Yup: Kz placed the poles of (Az - Bz Kz) where we wanted.

Step 3: Convert Kz to the gain times the state variables (X)

```
>> Kx = Kz*inv(T)

    3.0000    5.0000    7.0000    8.0000

>> eig(A - B*Kx)

   -4.0000
   -3.0000
   -2.0000
   -1.0000
```

The control law which places the closed-loop poles at { -1, -2, -3, -4 } is

$$U = -K_x X$$

where

$$K_x = \left[\begin{array}{cccc} 3 & 5 & 7 & 8 \end{array}\right]$$

Example 2: Complex Poles

With pole placement, you can place the closed-loop poles anywhere. For example, find the feedback gain, Kx, which places the closed-loop poles at

$$\{ -1 + j3, \; -1 - j3, \; -5 + j2, \; -5 - j2 \}$$

Following the previous design...

Step 0: Input the system (done)

Step 1: Find the similarity transform which takes you to controller canonical form (done)

Step 2: Find the feedback gains, Kz, which places the closed-loop poles of the closed-loop system

```
>> Pd = poly([-1 + j*3,  -1 - j*3,  -5 + j*2,  -5-j*2])

     1    12    59   158   290

>> P = poly(eig(A))

    1.0000    7.0000   15.0000   10.0000    1.0000

>> dP = Pd - P

     0     5    44   148   289

>> Kz = dP([5,4,3,2])

   289   148    44     5
```

Checking Kz:

```
>> eig(Az - Bz*Kz)

  -5.0000 + 2.0000i
  -5.0000 - 2.0000i
  -1.0000 + 3.0000i
  -1.0000 - 3.0000i
```

Yes, Kz places the poles of (Az - Bz Kz) where we want.

Step 3: Convert Kz to Kx:

```
>> Kx = Kz*inv(T)

    5.0000   19.0000   61.0000  204.0000
```

Check Kx:

```
>> eig(A - B*Kx)
```

```
-5.0000 + 2.0000i
-5.0000 - 2.0000i
-1.0000 + 3.0000i
-1.0000 - 3.0000i
```

Done. A control law which place the closed-loop poles at { -1 + j3, -1 - j3, -5 + j2, -5 - j2 } is

$$U = -K_x X$$

$$K_x = \left[\begin{array}{cccc} 5 & 19 & 61 & 204 \end{array} \right]$$