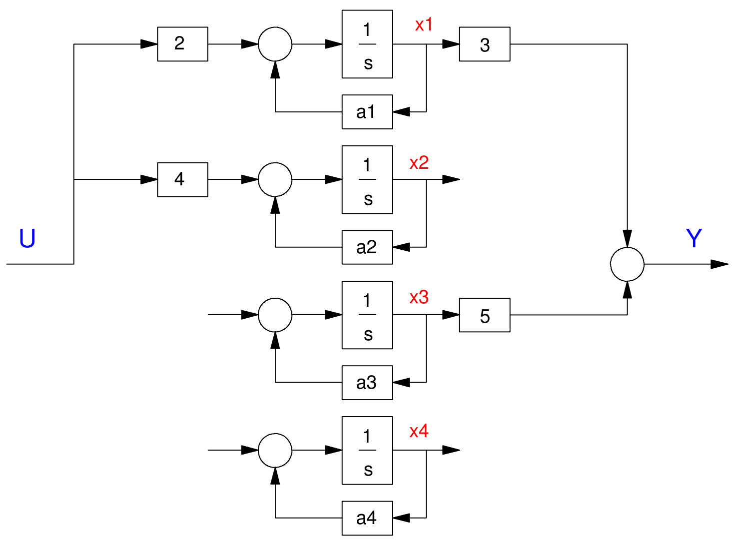
Controllability and Observability

Controllability and Observability are properties of systems which relate to whether the states can be driven to any arbitrary state from a given input (controllable) or whether you can deduce what the system's states are from a given output (observabilty). Controllability and Observability are easiest to see in Jordan form. Consider the following system:

$$sX = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_f \end{bmatrix} X + \begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 3 & 0 & 5 & 0 \end{bmatrix} X$$

The block diagram for this system is



Block Diagram Representation

From the block diagram, it is clear that the input cannot effect states X3 and X4. These states are termed uncontrollable. The output Y cannot see states X2 and X4 - meaning no amount of filtering or other mathematical trickery can deduce the value of X2 and X4 based upon measurements of Y. Likewise, states X2 and X4 are unobservable:

- x1 is both controllable and observable
- x2 is controllable but not observable
- x3 is observable but not controllable
- x4 is neither controllable nor observable

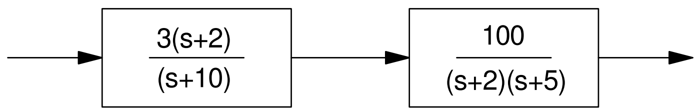
One way to tell if a system is uncontrollable or unobservable is to count the number of states in the transfer function. The above system is 4th-order. The transfer function from U to Y, however, is 1st-order:

$$Y = \left(\frac{6}{s+a_1}\right)U$$

The missing three states tells you that three states are uncontrollable, unobservable, or both. (Corollary: if the transfer function for an Nth-order system is Nth-order, the system is controllable and observable). The following are tests to determine if a system is uncontrollable or unobservable.

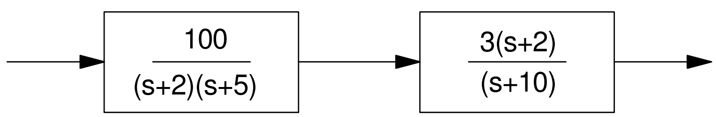
Pole Zero Cancellation

If a zero which cancels a pole comes before the pole, that state is uncontrollable.



The pole at -2 is uncontrollable due to zero canceling the pole before the pole

If a zero which cancels a pole comes after the pole, that state is unobservable.



The pole at -2 is unobservable due to zero canceling the pole after the pole

Controllability Matrix:

Theorem: If

$$rank\left(\begin{bmatrix} B & AB & A^2B & \dots & A^{N-1}B \end{bmatrix}\right) = N$$

the system is controllable.

Proof: This is easier to see in discrete time. Assume you have a 4th-order discrete-time system:

$$zX = AX + BU$$

If the input, U, can drive the system to any arbitrary state, then the system is controllable. Assume X(0) = 0. X at any given time is then

$$X(0) = X_0$$

$$X_1 = AX_0 + BU_0$$

$$X_2 = AX_1 + BU_1 = A^2X_0 + ABU_0 + BU_1$$

$$X_3 = AX_2 + BU_2 = A^3X_0 + A^2BU_0 + ABU_1 + BU_2$$

$$X_4 = AX_3 + BU_3 = A^4X_0 + A^3BU_0 + A^2BU_1 + ABU_2 + BU_3$$

or in matrix form

$$X_4 - A^4X_0 = \left[\begin{array}{cccc} B & AB & A^2B & A^3B \end{array}\right]\left[\begin{array}{c} U_3 \\ U_2 \\ U_1 \\ U_0 \end{array}\right]$$

If the matrix (termed the *controllability matrix)*

$$\left[\begin{array}{cccc} B & AB & A^2B & A^3B \end{array}\right]$$

is full rank (i.e. it is invertable), then you can solve for U0..U3. This means that you can drive the system to any state at k=4, regardless of the initial condition (i.e. the system is controllable).

You might think you need to keep going, adding more and more terms. Actually, if you have an Nth order system, you can stop after N terms in the controllability matrix. This is due to the Calley Hamilton theorem.

The Calley Hamilton theorem states that any matrix satisfies its own characteristic equation. (i.e. denominator polynomial.) You can see this in matlab:

Take for example an arbitrary 4x4 matrix, A. Find it's characteristic polynomial

```
>> A = [-2,1,0,0;1,-2,1,0;0,1,-2,1;0,0,1,-1]

    -2     1     0     0
     1    -2     1     0
     0     1    -2     1
     0     0     1    -1

>> P = poly(eig(A))

P =

    1.0000    7.0000   15.0000   10.0000    1.0000
```

$$s^4 + 7s^3 + 15s^2 + 10s + 1 = 0$$

The Calley Hamilton theorem states that any matrix satisfies its own characteristic equtation:

$$A^4 + 7A^3 + 15A^2 + 10A + I = 0$$

Checking in Matlab:

```
>> A^4 + 7*A^3 + 15*A^2 + 10*A + 1*A^0

     0     0     0     0
     0     0     0     0
     0     0     0     0
     0     0     0     0
```

What this tells you is that A4 is linearly dependent upon {A3, A2, A, I}. Likewise, there is no need to add A4B or higher order terms to the controllability matrix: they will not increase the rank of it.

If the controllability matrix is full rank, then you can drive a discrete-time system to any state at time k = N. The same holds for continuous time systems: if the controllability matrix is full rank, you can drive the states to any value in finite time.

For systems like a cart and pendulum, it means it's possible to stabilize the system.

Observability matrix

The corollary to controllability is oservability.

A system is said to be observable if you can deduce what the states are (x1, x2, x3, ...) from the output and its derivatives. This will be important later on in the course when we start designing state estimators:

- If a system is observable, you can determine the systems states from the system's output
- If a system is not observable, then at least one state does not contribute to the output you're looking at. This means there is no information about that state in the output and you cannot determine the value of that state.

For example, consider the system with U=0

$$sX = AX + BU$$

$$y = CX$$

Can you determine X using only the output, y, and its derivatives?

The derivatives of y are

$$sy = s(CX) = CAX$$

$$s^2 y = s(CAX) = CA^2 X$$

$$s^3 y = s(CA^2 X) = CA^3 X$$

Putting this in matrix form

$$\begin{bmatrix} y \\ sy \\ s^2y \\ s^3y \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} X$$

If

$$rank\left(\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}\right) = 4$$

then you can solve for X given only the output, y, and its derivatives. This means the system is observable.

Again, the Calley Hamilton theorem tells you that for an Nth order system, there is no need to go beyond N-1 when forming the observably matrix.

Observability Matrix: $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N-1} \end{bmatrix}$

Rank, Determinant, and Eigenvalues

There are several ways to determine if the controllability / observability matrices are full rank in Matlab.

Rank(): The Matlab command *rank* determines how many linearly independent columns a matrix has. This is a Boolean test: each eigenvector either is or is not excited.

Example: 4-Stage RC filter

```
>> A = [-2,1,0,0;1,-2,1,0;0,1,-2,1;0,0,1,-1]

    -2     1     0     0
     1    -2     1     0
     0     1    -2     1
     0     0     1    -1
```

The eigenvectors are:

```
>> [M,V] = eig(A)

M

   -0.4285   -0.6565    0.5774    0.2280
    0.6565    0.2280    0.5774    0.4285
   -0.5774    0.5774   -0.0000    0.5774
    0.2280   -0.4285   -0.5774    0.6565
```

If the B matrix includes all four eigenvetors, then the controllability matrix is full rank:

```
B = M(:,1) + M(:,2) + M(:,3) + M(:,4)
AB = [B, A*B, A*A*B, A*A*A*B]
rank(AB)

ans =     4
```

If the B matrix only includes 3 of the 4 eigenvectors, the rank is only 3:

```
B = M(:,1) + M(:,2) + M(:,3);
AB = [B, A*B, A*A*B, A*A*A*B];
rank(AB)

ans =     3
```

Note: this is a binary output: anything non-zero shows up as being full rank

```
B = M(:,1) + M(:,2) + M(:,3) + 0.001*M(:,4);
AB = [B, A*B, A*A*B, A*A*A*B];
rank(AB)

ans =     4
```

Eigenvalues: A second way to determine if the controllability / observability matrix is full rank is to check its eigenvalues:

- If all eigenvalues are non-zero, the system is controllable / observable.
- If an eigenvalue is *close* to zero, the system is weakly controllable / observable.

An eigenvalue close to zero tells you that one or modes is very difficult to control from the input.

Note that

- If a matrix is not full rank, its determinant is zero.
- The trace is equal to the sum of the eigenvalues
- The determinant is equal to the product of the eigenvalues
- If the determinant is zero, at least one eigenvalue must also be zero.

An eigenvalue (or determinant) close to zero implies that the system is *weakly controllable*. In theory, you can drive the system to any state. In practice, the inputs required to do so may be excessive.

Example: Consider the 4-stage RC filter from before along with the A matrix and its corresponding eigenvectors. If the B matrix incorporates all four eigenvectors, then all eigenvalues are far from zero and the system is controllable:

```
A = [-2,1,0,0;1,-2,1,0;0,1,-2,1;0,0,1,-1];
B = M(:,1) + M(:,2) + M(:,3) + M(:,4);
AB = [B, A*B, A*A*B, A*A*A*B];

rank(AB)
   ans =     4
```

```
eig(AB)
  -6.0649 + 6.0095i
  -6.0649 - 6.0095i
   0.2123 + 0.5704i
   0.2123 - 0.5704i

det(AB)

ans =    27.0000
```

If one of the eigenvectors has a weighting of zero, one eigenvalue is zero and the system is uncontrollable:

```
B = M(:,1) + M(:,2) + M(:,3) + 0*M(:,4);
AB = [B, A*B, A*A*B, A*A*A*B];

rank(AB)
   ans =  3

det(AB)
   ans = -4.7138e-015

eig(AB)
  -6.2182 + 7.3686i
  -6.2182 - 7.3686i
   0.5474
   0.0000
```

If one of the eigenvectors has a weighting which is close to zero, the system is weakly controllable:

```
B = M(:,1) + M(:,2) + M(:,3) + 0.01*M(:,4);
AB = [B, A*B, A*A*B, A*A*A*B];

rank(AB)
   ans =     4

det(AB)
   ans =    0.0270

eig(AB)
  -6.2167 + 7.3564i
  -6.2167 - 7.3564i
   0.5410
   0.0054
```

PBH Rank Test

The controllability / observability matrices tell you if the system is controllable or observable. One problem with this test is it doesn't tell you *which* mode is not controllable or observable. The P.B.H. rank test does both:

- It tells you which modes are controllable / observable, and
- Which ones are not.

PBH Test for controllability

A system is controllable if

$$\rho[A - \lambda I, B] = N$$

for all λ.

PBH Test for Observability:

A system is observable if

$$\rho\left(\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}\right) = N$$

for all λ.

Note: $[A - \lambda I]$ is full rank everywhere *except* when λ is an eigenvalue of A. Likewise, you only need to check at the eigenvalues.

If the rank is *not* N, then that mode is uncontrollable / unobservable.

Example: 4-stage RC filter where B only includes 3 eigenvectors:

```
A = [-2,1,0,0;1,-2,1,0;0,1,-2,1;0,0,1,-1];
B = M(:,1) + M(:,2) + M(:,3);
P = eig(A)

   -3.5321
   -2.3473
   -1.0000
   -0.1206
```

Check which eigenvalue is controllable:

```
rank([A - P(1)*eye(4,4), B])
ans =     4

rank([A - P(2)*eye(4,4), B])
ans =     4
```

```
rank([A - P(3)*eye(4,4), B])
ans =      4

rank([A - P(4)*eye(4,4), B])
ans =      3
```

The 4th eigenvalue (s = -0.1206) is not controllable.

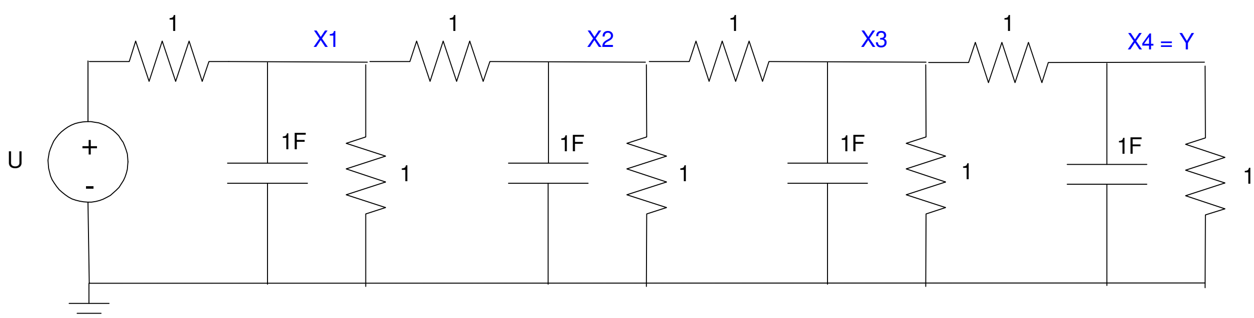
This also works with the C matrix and observabilty:

```
Mi = inv(M);
C = Mi(1,:) + Mi(2,:) + Mi(3,:);

rank([ C ; A - P(1)*eye(4,4) ] )
ans =      4

rank([ C ; A - P(2)*eye(4,4) ] )
ans =      4

rank([ C ; A - P(3)*eye(4,4) ] )
ans =      4

rank([ C ; A - P(4)*eye(4,4) ] )
ans =      3
```

The 4th eigenvalue (s = -0.1206) is not observable.

Example 1: RC Filter

Is the following system controllable and observable? (heat equation with four states)



4-Stage RC filter

The dynamics are

$$sX = \begin{bmatrix} -3 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} U$$

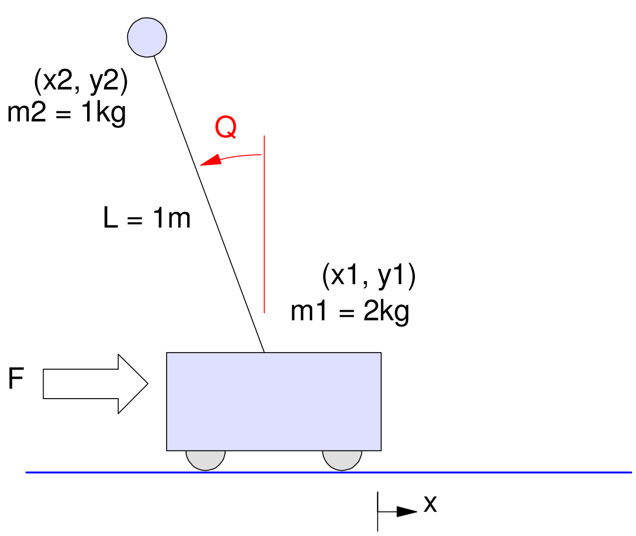
$$Y = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} X$$

In Matlab:

```
A = [-3,1,0,0 ; 1,-3,1,0 ; 0,1,-3,1 ; 0,0,1,-2]

    -3      1      0      0
     1     -3      1      0
     0      1     -3      1
     0      0      1     -2

B = [1;0;0;0]

     1
     0
     0
     0

C = [0,0,0,1]

     0      0      0      1

rank([B, A*B, A^2*B, A^3*B])

    ans =      4

rank([C ; C*A ; C*A^2 ; C*A^3])

    ans =      4
```

Both the controllability matrix and the observability matrix are full rank. This system is both controllable and observable.

Example 2: The dynamics for a cart and pendulum are



Cart and Pendulum System from Lecture #7

The dynamics are:

$$s\begin{bmatrix} x \\ \theta \\ x' \\ \theta' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -4.9 & 0 & 0 \\ 0 & 14.7 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \theta \\ x' \\ \theta' \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ -0.5 \end{bmatrix}F + \begin{bmatrix} 0 \\ 0 \\ -0.5 \\ 1.5 \end{bmatrix}T$$

where F is the force on the cart and T is the torque on the beam.

Is this system controllable from F?

```
A = [0,0,1,0;0,0,0,1;0,-4.9,0,0;0,14.7,0,0]

         0         0    1.0000         0
         0         0         0    1.0000
         0   -4.9000         0         0
         0   14.7000         0         0

BF = [0;0;0.5;-0.5]

         0
         0
    0.5000
   -0.5000

rank([BF, A*BF, A^2*BF, A^3*BF])

   ans =     4
```

Yes, this is controllable from the force input. It is possible to balance a yardstick on your hand.

If you look at the eigenvalues, you get an idea of how controllable the system is

```
eig([B, A*B, A^2*B, A^3*B])

   1.5652
   0.9750 + 1.2245i
   0.9750 - 1.2245i
  -1.5652
```

All four eigenvalues are about the same magnitude. It shouldn't be that hard to control a cart and pendulum system.

Is this system controllable from the torque, T?

```
B = [0;0;0.5;-1.5]

         0
         0
    0.5000
   -1.5000
```

```
rank([B, A*B, A^2*B, A^3*B])

    ans =     2
```

No, only two modes are controllable from the torque input.

Which modes are not controllable?

Use the PBH test:

```
P = eig(A)

         0
         0
    3.8341
   -3.8341

rank([B, A-P(1)*eye(4,4)])
    ans =     3

rank([B, A-P(2)*eye(4,4)])
    ans =     3

rank([B, A-P(3)*eye(4,4)])
    ans =     4

rank([B, A-P(4)*eye(4,4)])
    ans =     4
```

The poles at s = 0 are uncontrollable. These correspond to the cart position and velocity (found from the eigenvectors).

Is the cart observable from the cart position?

```
C = [1,0,0,0];
rank([C ; C*A ; C*A^2 ; C*A^3])
   ans =     4
```

Yes, surprisingly I can determine the angle of the pendulum and its velocity just by measuring position. It doesn't seem like you should be able to do this, but the math tells you that you can.

If you look at the eigenvalues, it's actually fairly easy to determine the beam angle just by measuring the position. This shows up with all four eigenvalues being reasonably far from zero

```
eig([C ; C*A ; C*A^2 ; C*A^3])

        0 + 2.2136i
        0 - 2.2136i
   1.0000
  -4.9000
```

Is the cart observable from beam angle?

```
C = [0,1,0,0];
rank([C ; C*A ; C*A^2 ; C*A^3])
   ans =     2
```

No, there are two modes you cannot see. From the PBH test, these are the poles at s = 0 (position and velocity)

```
rank([C ; A - P(1)*eye(4,4)])
   ans =     3

rank([C ; A - P(2)*eye(4,4)])
   ans =     3

rank([C ; A - P(3)*eye(4,4)])
   ans =     4

rank([C ; A - P(4)*eye(4,4)])
   ans =     4
```