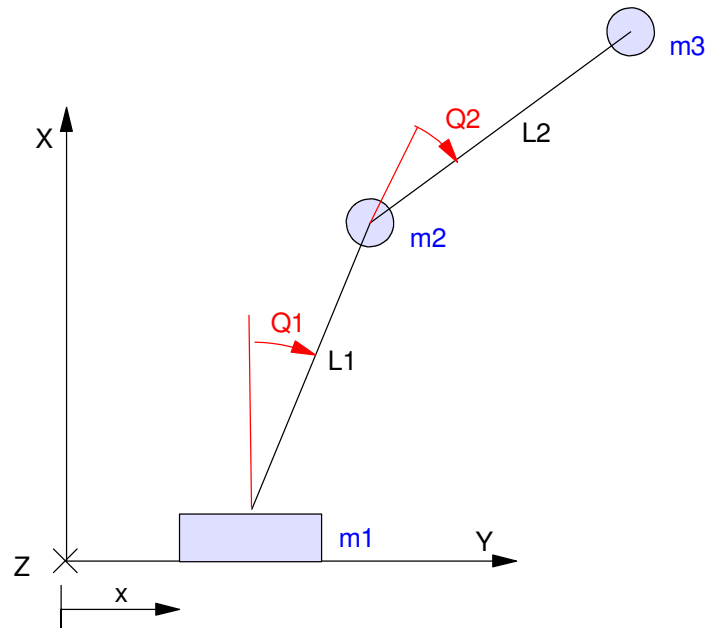


Dynamics of a Double Pendulum

A Gantry system with a second mass added to it is shown below. Assume

- $m_1 = m_2 = m_3 = 1\text{kg}$
- $l_1 = l_2 = 1\text{m}$



Determine the position of each mass as a function of x , θ_1 , and θ_2

Mass 1:

$$y_1 = x$$

$$x_1 = 0$$

Mass 2:

$$y_2 = y_1 + l_1 \sin(\theta_1)$$

$$x_2 = x_1 + l_1 \cos(\theta_1)$$

Mass 3:

$$y_3 = y_2 + l_2 \sin(\theta_1 + \theta_2)$$

$$x_3 = x_2 + l_2 \cos(\theta_1 + \theta_2)$$

Simplifying and using shorthand notation

$$y_1 = x$$

$$x_1 = 0$$

$$y_2 = x + s_1$$

$$x_2 = c_1$$

$$y_3 = x + s_1 + s_{12}$$

$$x_3 = c_1 + c_{12}$$

Determine the kinetic energy of the system

Take the derivatives:

$$\dot{y}_1 = \dot{x}$$

$$\dot{x}_1 = 0$$

$$\dot{y}_2 = \dot{x} + c_1 \dot{\theta}_1$$

$$\dot{x}_2 = -s_1 \dot{\theta}_1$$

$$\dot{y}_3 = \dot{x} + c_1 \dot{\theta}_1 + c_{12} (\dot{\theta}_1 + \dot{\theta}_2) \quad \dot{x}_3 = -s_1 \dot{\theta}_1 - s_{12} (\dot{\theta}_1 + \dot{\theta}_2)$$

$$KE = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2} m_3 (\dot{x}_3^2 + \dot{y}_3^2)$$

$$KE = \frac{1}{2} (\dot{x}^2) + \frac{1}{2} \left((\dot{x} + c_1 \dot{\theta}_1)^2 + (-s_1 \dot{\theta}_1)^2 \right) + \frac{1}{2} \left((\dot{x} + c_1 \dot{\theta}_1 + c_{12} (\dot{\theta}_1 + \dot{\theta}_2))^2 + (-s_1 \dot{\theta}_1 - s_{12} (\dot{\theta}_1 + \dot{\theta}_2))^2 \right)$$

Simplifying

$$KE = \frac{3}{2} \dot{x}^2 + \dot{\theta}_1^2 + \frac{1}{2} (\dot{\theta}_1 + \dot{\theta}_2)^2 + 2c_1 \dot{x} \dot{\theta}_1 + c_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) + c_{12} \dot{x} (\dot{\theta}_1 + \dot{\theta}_2)$$

Determine the potential energy of the system

$$PE = m_1 g x_1 + m_2 g x_2 + m_3 g x_3$$

$$PE = g c_1 + g (c_1 + c_{12})$$

$$PE = 2g c_1 + g c_{12}$$

Force on the Mass #1:

Form the LaGrangian:

$$L = \frac{3}{2}\dot{x}^2 + \dot{\theta}_1^2 + \frac{1}{2}(\dot{\theta}_1 + \dot{\theta}_2)^2 + 2c_1\dot{x}\dot{\theta}_1 + c_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2) + c_{12}\dot{x}(\dot{\theta}_1 + \dot{\theta}_2) - 2gc_1 - gc_{12}$$

Determine the force on the mass:

$$F = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x}$$

$$F = \frac{d}{dt}\left(3\dot{x} + 2c_1\dot{\theta}_1 + c_{12}(\dot{\theta}_1 + \dot{\theta}_2)\right) - 0$$

$$F = 3\ddot{x} + 2c_1\ddot{\theta}_1 - 2s_1\dot{\theta}_1^2 + c_{12}(\ddot{\theta}_1 + \ddot{\theta}_2) - s_{12}(\dot{\theta}_1 + \dot{\theta}_2)^2$$

Torque on Q1

$$L = \frac{3}{2}\dot{x}^2 + \dot{\theta}_1^2 + \frac{1}{2}(\dot{\theta}_1 + \dot{\theta}_2)^2 + 2c_1\dot{x}\dot{\theta}_1 + c_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2) + c_{12}\dot{x}(\dot{\theta}_1 + \dot{\theta}_2) - 2gc_1 - gc_{12}$$

$$T_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1}$$

$$T_1 = \frac{d}{dt}\left(2\dot{\theta}_1 + (\dot{\theta}_1 + \dot{\theta}_2) + 2c_1\dot{x} + c_2(\dot{\theta}_1 + \dot{\theta}_2) + c_2\dot{\theta}_1 + c_{12}\dot{x}\right) - (-2s_1\dot{x}\dot{\theta}_1 - s_{12}\dot{x}(\dot{\theta}_1 + \dot{\theta}_2) + 2gs_1 + gs_{12})$$

$$T_1 = 2\ddot{\theta}_1 + (\ddot{\theta}_1 + \ddot{\theta}_2) - 2s_1\dot{x}\dot{\theta}_1 + 2c_1\ddot{x} - s_2(\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_2 + c_2(\ddot{\theta}_1 + \ddot{\theta}_2) - s_2\dot{\theta}_1\dot{\theta}_2 + c_2\ddot{\theta}_1 - s_{12}\dot{x}(\ddot{\theta}_1 + \ddot{\theta}_2) + c_{12}\ddot{x} + 2s_1\dot{x}\dot{\theta}_1 + s_{12}\dot{x}(\dot{\theta}_1 + \dot{\theta}_2) - 2gs_1 - gs_{12}$$

Torque on Q2

$$L = \frac{3}{2}\dot{\mathbf{x}}^2 + \dot{\theta}_1^2 + \frac{1}{2}(\dot{\theta}_1 + \dot{\theta}_2)^2 + 2c_1\dot{\mathbf{x}}\dot{\theta}_1 + c_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2) + c_{12}\dot{\mathbf{x}}(\dot{\theta}_1 + \dot{\theta}_2) - 2gc_1 - gc_{12}$$

$$T_2 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2}$$

$$T_2 = \frac{d}{dt}\left((\dot{\theta}_1 + \dot{\theta}_2) + c_2\dot{\theta}_1 + c_{12}\dot{\mathbf{x}}\right) - \left(-s_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2) - s_{12}\dot{\mathbf{x}}(\dot{\theta}_1 + \dot{\theta}_2) + gs_{12}\right)$$

$$T_2 = (\ddot{\theta}_1 + \ddot{\theta}_2) - s_2\dot{\theta}_1\dot{\theta}_2 + c_2\ddot{\theta}_1 - s_{12}\dot{\mathbf{x}}(\dot{\theta}_1 + \dot{\theta}_2) + c_{12}\ddot{\mathbf{x}} \\ + s_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2) + s_{12}\dot{\mathbf{x}}(\dot{\theta}_1 + \dot{\theta}_2) - gs_{12}$$

Nonlinear Dynamics of a Double Pendulum:

$$F = 3\ddot{\mathbf{x}} + 2c_1\ddot{\theta}_1 - 2s_1\dot{\theta}_1^2 + c_{12}(\ddot{\theta}_1 + \ddot{\theta}_2) - s_{12}(\dot{\theta}_1 + \dot{\theta}_2)^2$$

$$T_1 = 2\ddot{\theta}_1 + (\ddot{\theta}_1 + \ddot{\theta}_2) - 2s_1\dot{\mathbf{x}}\dot{\theta}_1 + 2c_1\ddot{\mathbf{x}} - s_2(\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_2 \\ + c_2(\ddot{\theta}_1 + \ddot{\theta}_2) - s_2\dot{\theta}_1\dot{\theta}_2 + c_2\ddot{\theta}_1 - s_{12}\dot{\mathbf{x}}(\dot{\theta}_1 + \dot{\theta}_2) + c_{12}\ddot{\mathbf{x}} \\ + 2s_1\dot{\mathbf{x}}\dot{\theta}_1 + s_{12}\dot{\mathbf{x}}(\dot{\theta}_1 + \dot{\theta}_2) - 2gs_1 - gs_{12}$$

$$T_2 = (\ddot{\theta}_1 + \ddot{\theta}_2) - s_2\dot{\theta}_1\dot{\theta}_2 + c_2\ddot{\theta}_1 - s_{12}\dot{\mathbf{x}}(\dot{\theta}_1 + \dot{\theta}_2) + c_{12}\ddot{\mathbf{x}} \\ + s_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2) + s_{12}\dot{\mathbf{x}}(\dot{\theta}_1 + \dot{\theta}_2) - gs_{12}$$

Another way to write this is

$$\mathbf{M}\ddot{\boldsymbol{\theta}} = \mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \mathbf{G}(\boldsymbol{\theta}) + \mathbf{T}$$

where

- $\mathbf{M}$ is the mass (inertia) matrix
- $\mathbf{C}()$ are the coriolis forces
- $\mathbf{G}()$ is the gravity matrix, and
- $\mathbf{T}$ is the input (torques and forces).

Placing the dynamics in this form:

$$3\ddot{x} + 2c_1\ddot{\theta}_1 + c_{12}(\ddot{\theta}_1 + \ddot{\theta}_2) = F + 2s_1\dot{\theta}_1^2 + -s_{12}(\dot{\theta}_1 + \dot{\theta}_2)^2$$

$$\begin{aligned} 2\ddot{\theta}_1 + (\ddot{\theta}_1 + \ddot{\theta}_2) + 2c_1\ddot{x} + c_2(\ddot{\theta}_1 + \ddot{\theta}_2) + c_2\ddot{\theta}_1 + c_{12}\ddot{x} = & T_1 + 2s_1\dot{x}\dot{\theta}_1 + s_2(\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_2 \\ & + s_2\dot{\theta}_1\dot{\theta}_2 + s_{12}\dot{x}(\dot{\theta}_1 + \dot{\theta}_2) \\ & - 2s_1\dot{x}\dot{\theta}_1 - s_{12}\dot{x}(\dot{\theta}_1 + \dot{\theta}_2) + 2gs_1 + gs_{12} \end{aligned}$$

$$\begin{aligned} (\ddot{\theta}_1 + \ddot{\theta}_2) + c_2\ddot{\theta}_1 + c_{12}\ddot{x} = & T_2 + s_2\dot{\theta}_1\dot{\theta}_2 + s_{12}\dot{x}(\dot{\theta}_1 + \dot{\theta}_2) \\ & - s_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2) - s_{12}\dot{x}(\dot{\theta}_1 + \dot{\theta}_2) + gs_{12} \end{aligned}$$

Placing in matrix form:

$$\begin{bmatrix} 3 & (2c_1 + c_{12}) & c_{12} \\ (2c_1 + c_{12}) & (3 + 2c_2) & (1 + c_2) \\ c_{12} & (1 + c_2) & 1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} +2s_1\dot{\theta}_1^2 + -s_{12}(\dot{\theta}_1 + \dot{\theta}_2)^2 \\ +2s_1\dot{x}\dot{\theta}_1 + s_2(\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_2 + s_2\dot{\theta}_1\dot{\theta}_2 - 2s_1\dot{x}\dot{\theta}_1 \\ +s_2\dot{\theta}_1\dot{\theta}_2 - s_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2) \end{bmatrix} + g \begin{bmatrix} 0 \\ +2s_1 + s_{12} \\ s_{12} \end{bmatrix} + \begin{bmatrix} F \\ T_1 \\ T_2 \end{bmatrix}$$

Linearized Dynamics

For small angles

$$c_1 \approx 1$$

$$s_1 \approx \theta_1$$

$$\dot{\theta}_1\dot{\theta}_2 \approx 0$$

$$\begin{bmatrix} 3 & 3 & 1 \\ 3 & 5 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = g \begin{bmatrix} 0 \\ +2\theta_1 + \theta_1 + \theta_2 \\ \theta_1 + \theta_2 \end{bmatrix} + \begin{bmatrix} F \\ T_1 \\ T_2 \end{bmatrix}$$

If the only input is the force on the base

$$\begin{bmatrix} 3 & 3 & 1 \\ 3 & 5 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = g \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ \theta_1 \\ \theta_2 \end{bmatrix} + \begin{bmatrix} F \\ T_1 \\ T_2 \end{bmatrix}$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = g \begin{bmatrix} 0 & -2 & 0 \\ 0 & 3 & -1 \\ 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} x \\ \theta_1 \\ \theta_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} F$$

In State-Space form

$$s \begin{bmatrix} x \\ \theta_1 \\ \theta_2 \\ \dot{x} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2g & 0 & 0 & 0 & 0 \\ 0 & 3g & -g & 0 & 0 & 0 \\ 0 & -3g & 3g & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta_1 \\ \theta_2 \\ \dot{x} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} F$$

Checking the controllability of the system:

```
>> Z = zeros(3,3);
>> I = eye(3,3);
>> A = [Z,I ; inv(M)*G*9.8,Z]
```

```

0      0      0      1.0000      0      0
0      0      0      0      1.0000      0
0      0      0      0      0      1.0000
0 -19.6000 -0.0000      0      0      0
0  29.4000 -9.8000      0      0      0
0 -29.4000 29.4000      0      0      0
```

```
>> B = [0;0;0;1;-1;1];
```

```
>> rank([B,A*B,A^2*B,A^3*B,A^4*B,A^5*B])
```

Surprisingly, the system is controllable from the force input.

Checking the eigenvalues:

```
>> eig(A)

      0
      0
 -6.8099
 -3.5250
  6.8099
  3.5250
```

The open-loop system is unstable as it should be. If you turn it off, it will fall.

If you reverse gravity, this is a double gantry system

```
>> A = [Z,I ; -inv(M)*G*9.8,Z]

      0      0      0      1.0000      0      0
      0      0      0      0      1.0000      0
      0      0      0      0      0      1.0000
      0     19.6000      0.0000      0      0      0
      0    -29.4000      9.8000      0      0      0
      0     29.4000    -29.4000      0      0      0

>> B = [0;0;0;1;-1;1]

      0
      0
      0
      1
     -1
      1

>> eig(A)

      0
      0
 0.0000 + 6.8099i
 0.0000 - 6.8099i
-0.0000 + 3.5250i
-0.0000 - 3.5250i

>> rank([B,A*B,A^2*B,A^3*B,A^4*B,A^5*B])

ans =      6
```

This too is controllable from the force input.

