Homework #8: ECE 461/661

Root-Locus with Complex Poles & Zeros, Gain Compensation, Lead Compensation

Root Locus with Complex Poles & Zeros

Sketch the root locus plot for the following systems for 0 < k < infinity. Also plot the

- real axis loci, break away points, jw crossings (if any), asymptotes, and departure/approach angle

1) $$G(s) = \left(\frac{s+5}{s\left(s^2+2s+5\right)}\right) = \left(\frac{s+5}{s(s+1+j2)(s+1-j2)}\right)$$

Real Axis Loci: (0, -5)

Breakaway Points: none

jw Crossing: j2.887

Asymptotes:

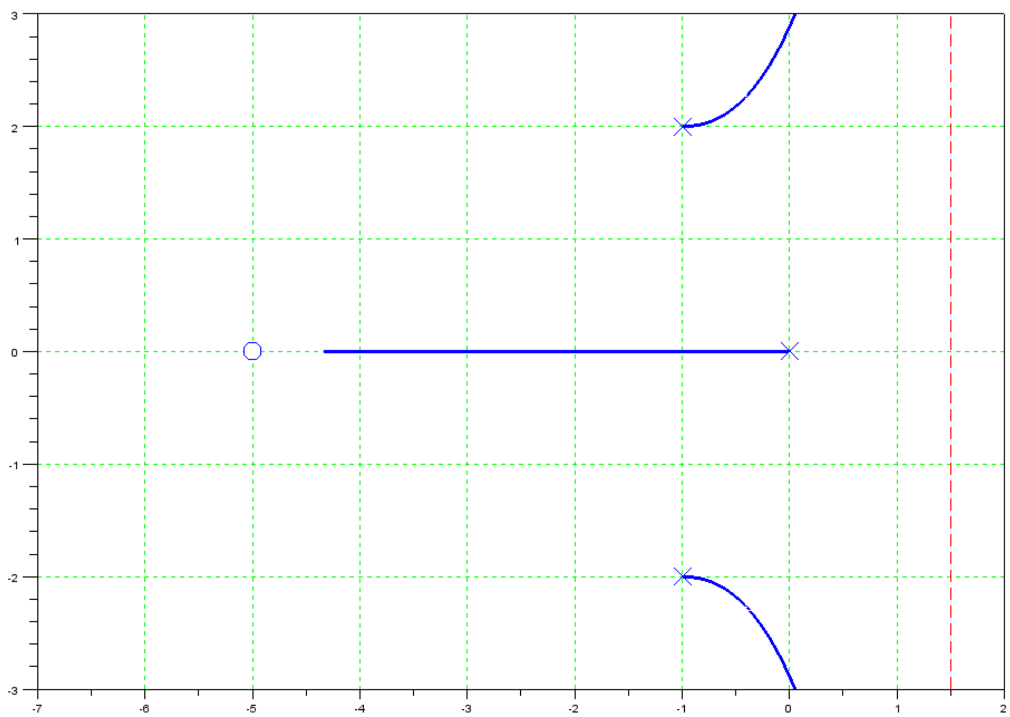
- 2 asymptotes 3 poles - 1 zero = 2 asymptotes
- Intersect: $\left(\frac{(0-1-1)-(-5)}{3-1}\right) = +1.5$
- Angle: +/- 90 degrees

Departure Angle: Analyze G(s) at a point just to the right of the pole

$$Angle\left(\frac{s+5}{s(s+1+j2)(s+1-j2)}\right)_{s=-1+j2} = 180^0$$

```
>> G = zpk([-5],[0,-1+2j,-1-2j],1)

     (s+5)
----------------
s (s^2 + 2s + 5)

>> s = -1+2j + 1e-6;
>> angle(evalfr(G,s)) * 180/pi

ans = -180.0000

>> q = ans + 180

q =  3.1513e-005
```

Departure angle = 0 degrees


3
2
1
0
-1
-2
-3
-7
-6
-5
-4
-3
-2
-1
0
1
2

2) $$G(s) = \left(\frac{s^2+2s+5}{s(s+2)(s+4)(s+6)}\right) = \left(\frac{(s+1+j2)(s+1-j2)}{s(s+2)(s+4)(s+6)}\right)$$

Real Axis Loci:	(0, -2), (-4, -6)
Breakaway Points:	-0.802, -5.064
jw Crossing	none

Asymptotes

- 2 asymptotes — 4 poles - 2 zeros
- Angle — +/- 90 degrees
- Intercept — -5.00

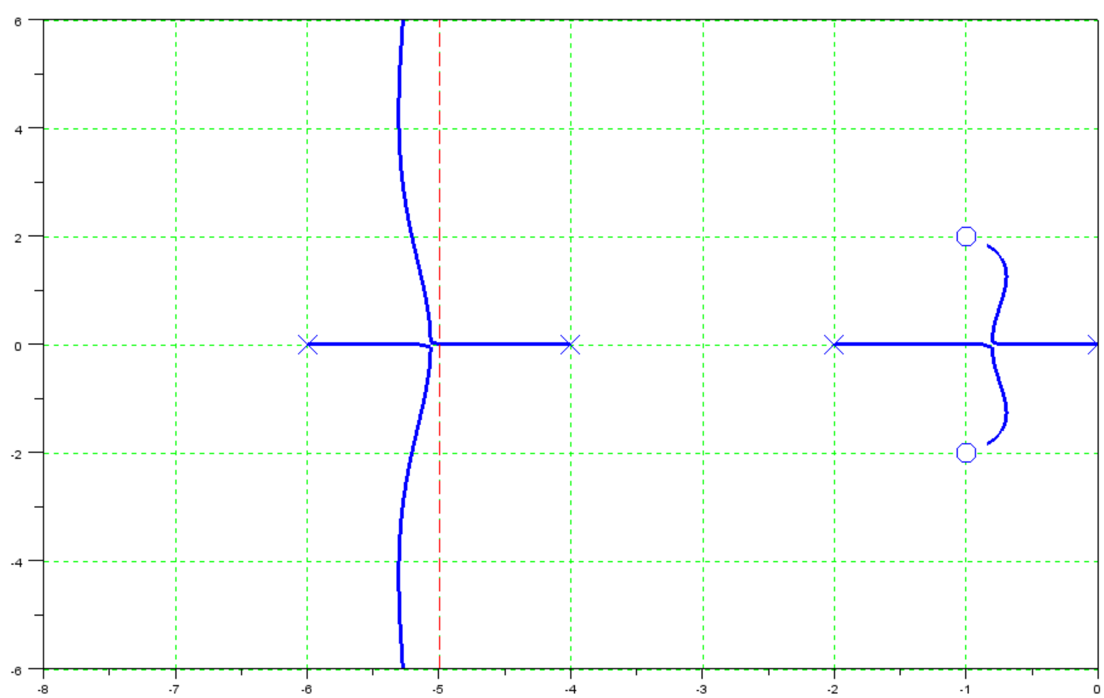
Approach Angle

$$angle\left(\frac{(s+1+j2)(s+1-j2)}{s(s+2)(s+4)(s+6)}\right)_{s\to -1+j2} = 180^0$$

Evaluage at epsilon to the right of the zero at -1 + j2

```
>> G = zpk([-1+2j,-1-2j],[0,-2,-4,-6],1)

   (s^2 + 2s + 5)
-------------------
s (s+2) (s+4) (s+6)

>> s = -1 + 2j + 1e-6;
>> angle(evalfr(G, s)) * 180/pi
ans = -145.4914

>> q = -180 - ans
q =  -34.5086
```

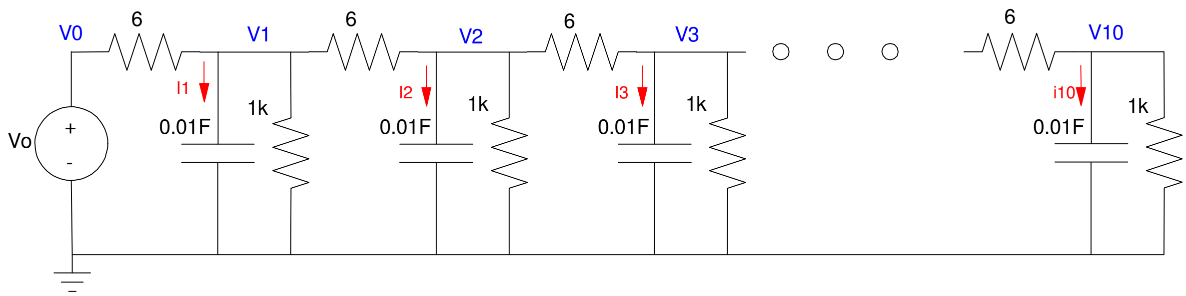
The approach angle is -34.5086 degrees



Gain & Lead Compensation

A 4th-order model for the following 10-stage RC filter is

$$G(s) = \left(\frac{170}{(s+0.47)(s+3.40)(s+9.00)(s+16.77)} \right)$$

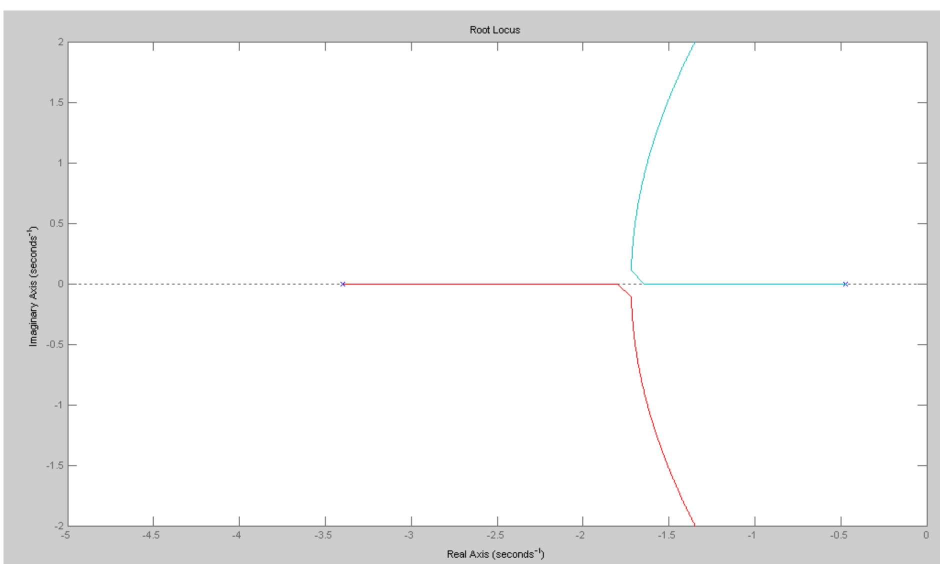


3) Design a gain compensator (K(s) = k) which results in

- The fastest system possible,
- With no overshoot for a step input (i.e. design for the breakaway point)

Step 1: Draw the root locus

```
>> G = zpk([],[-0.47,-3.40,-9.00,-16.77],170);
>> k = logspace(-2,2,1000)';
>> rlocus(G,k);
>> xlim([-5,0]);
>> ylim([-2,2]);
```



Step 2: Pick a point on the root locus. Since we want the fastest system with no overshoot, pick the breakaway point

s = -1.65 (approx)

Step 3: Find k at this point. At any point on the root locus

$$G \cdot k = -1$$

Pluging in numbers

$$\left(\frac{170}{(s+0.47)(s+3.40)(s+9.00)(s+16.77)} \right)_{s=-1.65} \cdot k = -1$$

In Matlab:

```
>> evalfr(G,-1.65)
   -0.7408

>> k = 1/abs(ans)

k =  1.3499
```

For this value of k, determine

a) The closed-loop dominant pole(s)

```
>> Gcl = minreal(G*k / (1 + G*k));
>> eig(Gcl)

   -1.6500
   -1.7929
   -9.5678
  -16.6293
```

This isn't *quite* the breakaway point since the other pole is at -1.7929 (not exactly equal to 1.65) but close

b) The 2% settling time,

$$T_s = \frac{4}{1.65} = 2.42 \text{ seconds}$$

c) The error constant, Kp, and

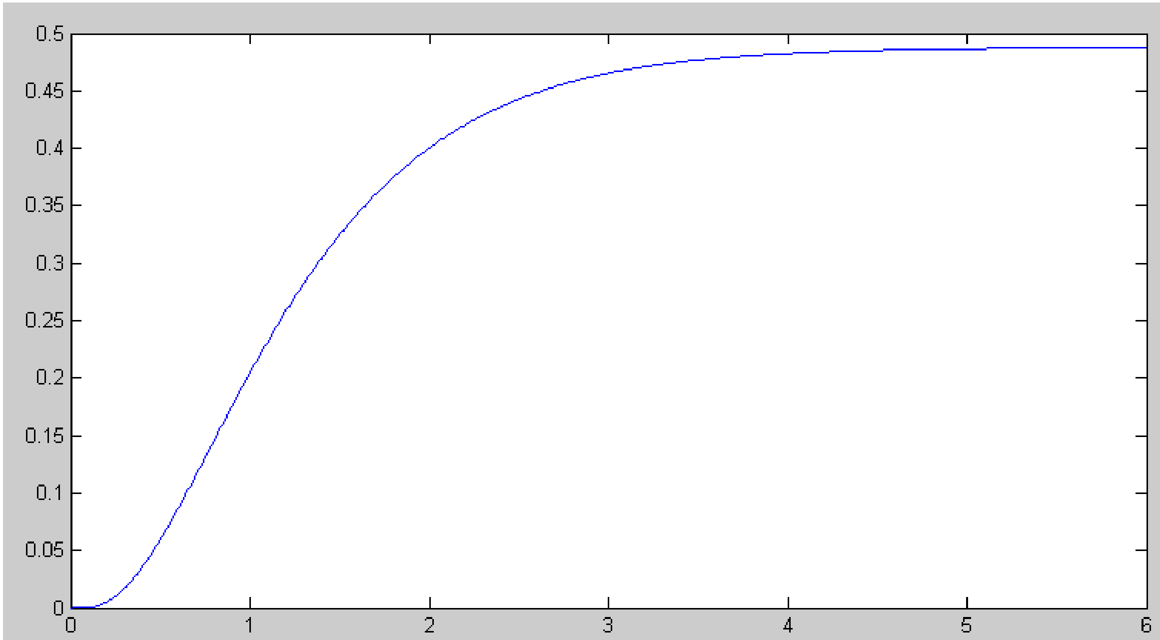
```
G = zpk([],[-0.47,-3.40,-9.00,-16.77],170);
>> k = 1 / abs(evalfr(G, -1.65))
k =    1.3499

>> Kp = evalfr(G,0) * k
Kp =    0.9515
```

The steady-state error for a step input.

```
>> Estep = 1 / (Kp + 1)
Estep =    0.5124
```

Check your design in Matlab or Simulink or VisSim

```
>> Gcl = minreal(G*k / (1+G*k));
>> t = [0:0.01:6]';
>> y = step(Gcl,t);
>> plot(t,y);;
```


0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
1
2
3
4
5
6

4) Design a gain compensator (K(s) = k) which results in 20% overshoot for a step input.

Step 1: Determine the damping line that corresponds to 20% overshoot

$$OS = 0.2 = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

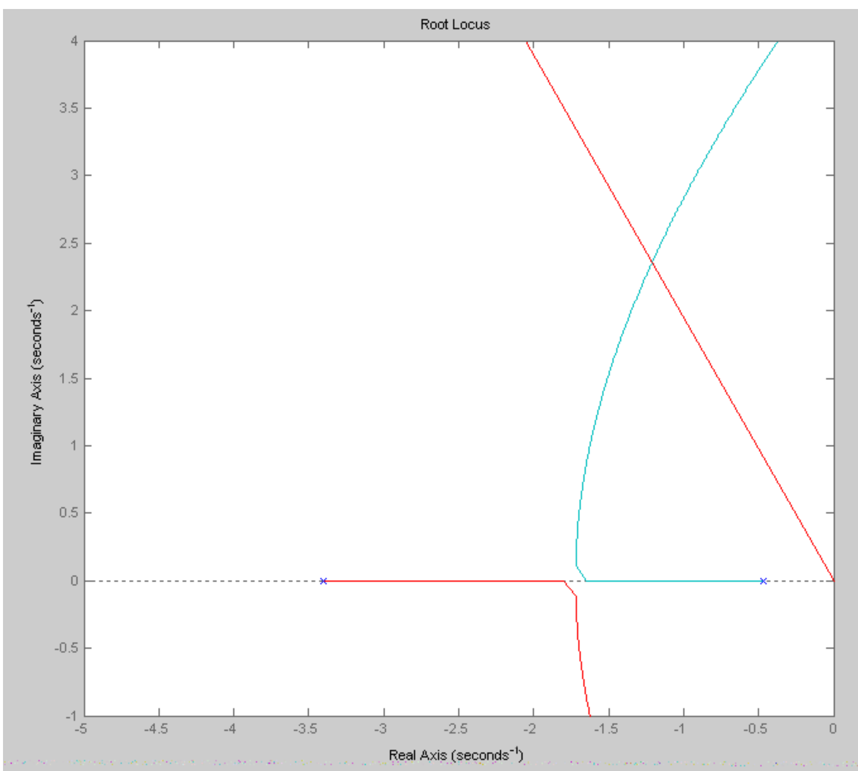
$$\zeta = 0.4560$$

$$\theta = \arccos(\zeta) = 62.61^0$$

Step 2: Sketch the root locus.

- Include the damping line
- Determine the point on the root locus that intersects the damping line

```
>> G = zpk([],[-0.47,-3.40,-9.00,-16.77],170);
>> k = logspace(-2,2,1000)';
>> rlocus(G,k);
>> hold on;
>> tan(acos(0.4560))
ans =     1.9517

>> plot([0,-3],[0,1.9517*3],'r')
>> xlim([-5,0]);
>> ylim([-1,4]);
```



From the graph (zooming in on the intersection)

```
>> s = -1.209 + j*2.36;
>> evalfr(G, -1.209 + j*2.36)
ans =  -0.1666 - 0.0000i
```

The complex part is almost zero. This is the correct point (or really close).

To find k, any point on the root locus satisfies

$$G \cdot k = -1$$

```
>> k = 1/abs(evalfr(G, -1.209 + j*2.36))

k =     6.0020
```

For this value of k, determine

a) The closed-loop dominant pole(s)

```
>> Gcl = minreal(G*k / (1+G*k));
>> eig(Gcl)

ans =

  -1.2092 + 2.3601i
  -1.2092 - 2.3601i
 -11.1897
 -16.0318
```

b)The 2% settling time,

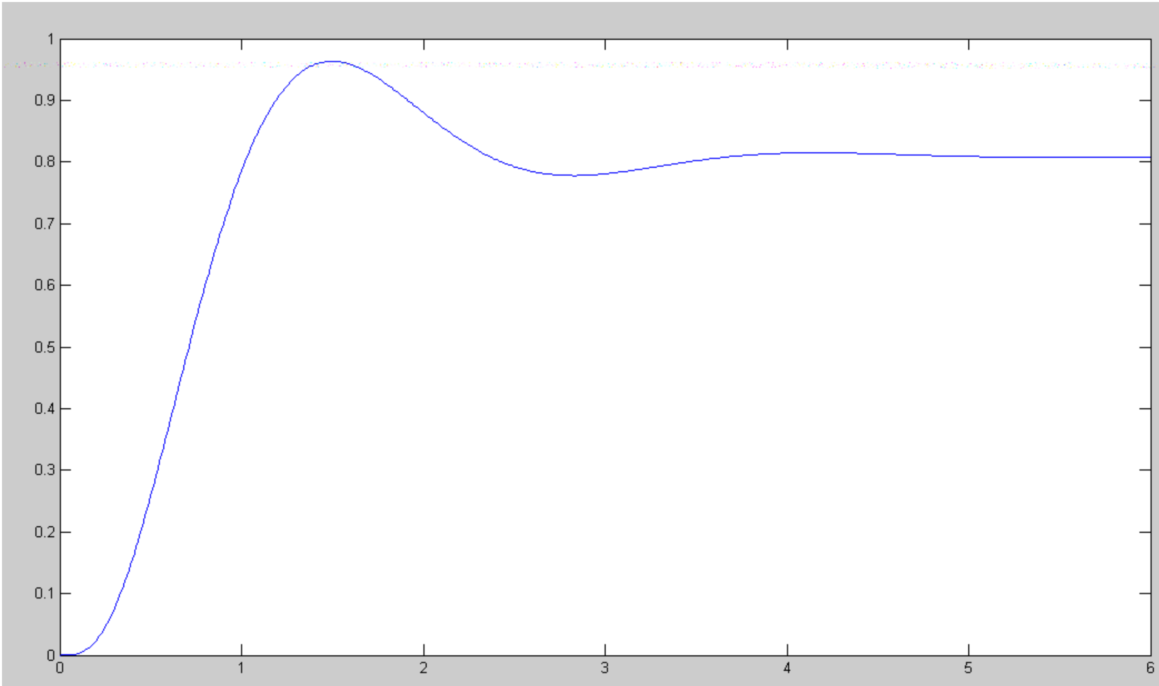
$$T_s = \frac{4}{1.2092} = 3.3080 \text{ seconds}$$

c)The error constant, Kp, and

```
>> Kp = evalfr(G,0) * k
Kp =     4.2305
```

d) The steady-state error for a step input.

```
>> Estep = 1 / (Kp + 1)
Estep =     0.1912

Check your design in Matlab or Simulink or VisSim>> t = [0:0.01:6]';
>> y = step(Gcl,t);
>> plot(t,y);
>> hold off
>> plot(t,y);
```



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1
2
3
4
5
6

5) Design a lead compensator, $K(s) = k\left(\frac{s+a}{s+10a}\right)$, which results in 20% overshoot for a step input.

$$G(s) = \left(\frac{170}{(s+0.47)(s+3.40)(s+9.00)(s+16.77)}\right)$$

Keep the pole at s = -0.47. This makes the system sort-of type-1. Cancel the next slowest pole

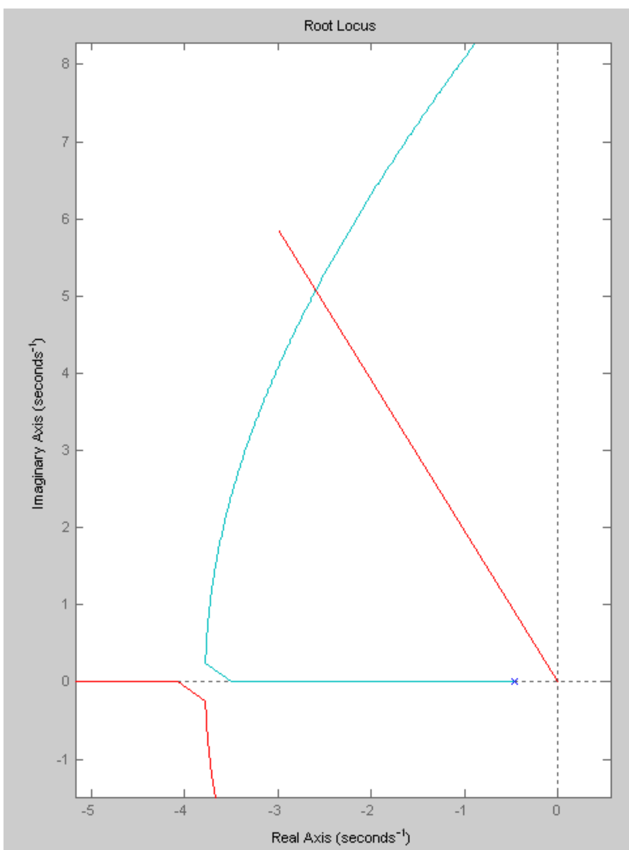
$$K(s) = k\left(\frac{s+3.40}{s+34}\right)$$

resulting in

$$GK = \left(\frac{170}{(s+0.47)(s+9.00)(s+16.77)(s+34)}\right)$$

Sketch the root locus along with the damping line

```
>> GK = zpk([],[-0.47,-34,-9.00,-16.77],170);
>> k = logspace(-2,2,1000)';
>> rlocus(GK,k);
>> k = logspace(-1,3,1000)';
>> rlocus(GK,k);
>> hold on
>> plot([0,-3],[0,1.9517*3],'r')
```



Zoom in to find the spot on the root locus

s = -2.5978 + j5.0701

Checking

```
>> s = -2.5978 + j*5.0701;
>> evalfr(GK,s)

ans =  -0.0079 - 0.0000i
```

The complex part is almost zero, so this point is on the root locus (or really close)

```
>> k = 1/abs(ans)

k =  126.4665
```

For this K(s), determine

a) The closed-loop dominant pole(s)

```
>> GKcl = minreal( (GK*k) / (1 + GK*k) );
>> eig(GKcl)

  -2.5978 + 5.0701i
 -22.9710
 -32.0734
  -2.5978 - 5.0701i
```

b) The 2% settling time,

$$T_s = \frac{4}{2.5978} = 1.5398 \text{ seconds}$$

c) The error constant, Kp, and

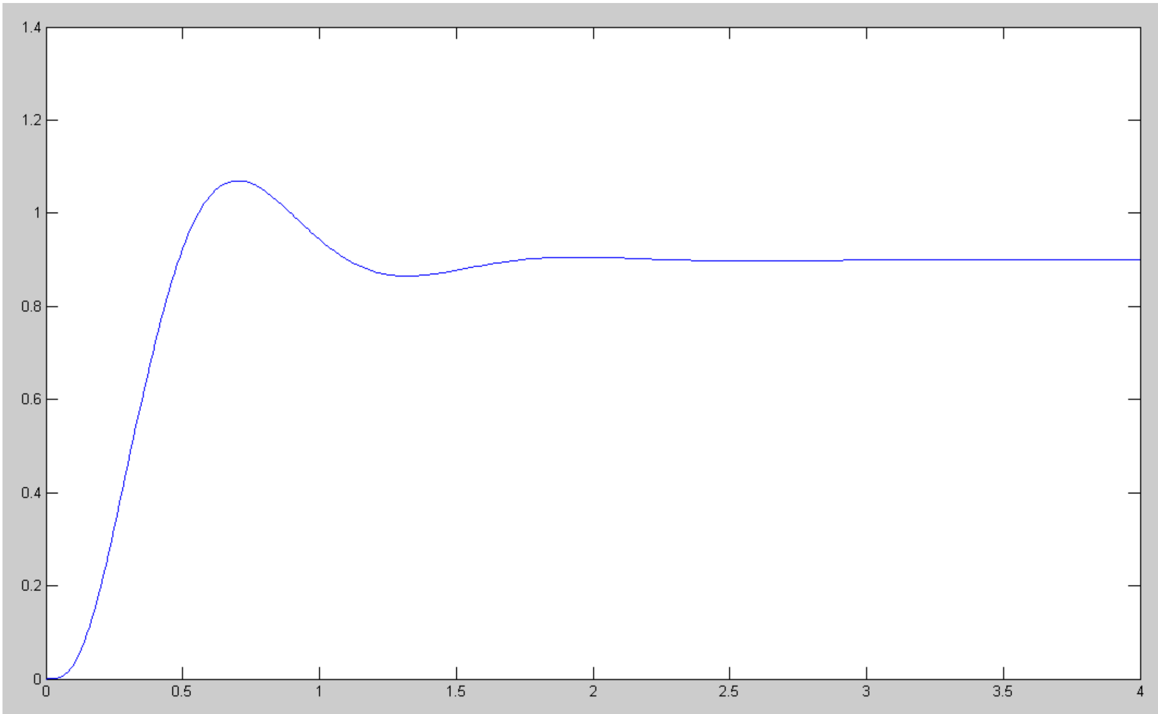
```
>> Kp = evalfr(GK,0) * k

Kp =    8.9140
```

d) The steady-state error for a step input.

```
>> Estep = 1 / (Kp + 1)

Estep =    0.1009
```

e) Check your design in Matlab or Simulink or VisSim

```
>> GKcl = minreal( (GK*k) / (1 + GK*k) );
>> t = [0:0.01:4]';
>> y = step(GKcl,t);
>> hold off
>> plot(t,y);
```



f) Give an op-amp circuit to implement K(s)

$$K(s) = 126.46\left(\frac{s+3.4}{s+34}\right)$$