

# ECE 376 - Homework #11

z-Transforms and Digital Filters.

1) Assume X and Y are related by the following transfer function

$$Y = \left( \frac{20(s+2)}{(s+1)(s+3)(s+10)} \right) X$$

a) What is the differential equation relating X and Y?

multiply out and cross multiply

$$Y = \left( \frac{20s+40}{s^3+14s^2+43s+30} \right) X$$

$$(s^3 + 14s^2 + 43s + 30)Y = (20s + 40)X$$

'sY' means 'the derivative of y', so (using prime notation for derivative):

$$y''' + 14y'' + 43y' + 30y = 20x' + 40x$$

$$\frac{d^3y}{dt^3} + 14\frac{d^2y}{dt^2} + 43\frac{dy}{dt} + 30y = 20\frac{dx}{dt} + 40x$$

b) Find y(t) assuming

$$x(t) = 2 + 4 \sin(3t)$$

Use superposition and treat this as two separate problems

$$x_1(t) = 2$$

$$s = 0$$

$$X = 2$$

$$Y = \left( \frac{20s+40}{s^3+14s^2+43s+30} \right)_{s=0} (2) = 2.667$$

$$y_1(t) = 2.667$$

$$x_2(t) = 4 \sin(3t)$$

$$s = j3$$

$$X = 0 - j4$$

*real = cosine, imag = -sine*

$$Y = \left( \frac{20s+40}{s^3+14s^2+43s+30} \right)_{s=j3} \cdot (0 - j4) = -2.0061 - j0.4648$$

$$y_2(t) = -2.0061 \cos(3t) + 0.4648 \sin(3t)$$

$$y(t) = y_1 + y_2 = 2.667 - 2.0061 \cos(3t) + 0.4648 \sin(3t)$$

2) Assume X and Y are related by the following transfer function

$$Y = \left( \frac{0.2(z+1)}{(z-0.98)(z-0.96)} \right) X$$

a) What is the difference equation relating X and Y?

multiply out and cross multiply

$$Y = \left( \frac{0.2z+0.2}{z^2-1.94z+0.9408} \right) X$$

$$(z^2 - 1.94z + 0.9408)Y = (0.2z + 0.2)X$$

'zY' means 'y(k+1)' or 'the next value of y(k)'

$$y(k+2) - 1.94y(k+1) + 0.9408y(k) = 0.2x(k+1) + 0.2x(k)$$

b) Find y(t) assuming a sampling rate of T = 0.01 second

$$x(t) = 2 + 3 \sin(4t)$$

Use superposition and treat this as two separate problems

$$x_1(t) = 2$$

$$s = 0$$

$$z = e^{sT} = 1$$

$$Y = \left( \frac{0.2(z+1)}{(z-0.98)(z-0.96)} \right)_{z=1} \cdot (2) = 1000$$

$$y_1(t) = 1000$$

$$x_2(t) = 3 \sin(4t)$$

$$s = j4$$

$$z = e^{sT} = e^{j0.04} = 1 \angle 2.292^\circ$$

$$X = 0 - j3 \quad \text{real} = \cosine, \text{ imag} = -sine$$

$$Y = \left( \frac{0.2(z+1)}{(z-0.98)(z-0.96)} \right)_{z=1 \angle 2.292^\circ} \cdot (0 - j3)$$

$$Y = -457.27 + j155.47$$

$$y_2(t) = -457.27 \cos(4t) - 155.47 \sin(4t)$$

The total answer is the sum of the two

$$y(t) = 1000 - 457.27 \cos(4t) - 155.47 \sin(4t)$$

Problem 3) Assume  $G(s)$  is a low-pass filter with real poles:

$$G(s) = \left( \frac{1000}{(s+3)(s+7)(s+20)} \right)$$

3) Design a digital filter,  $G(z)$ , which has approximately the same gain vs. frequency as  $G(s)$ . Assume a sampling rate of  $T = 0.01$  second.

Plot the gain vs. frequency for both filters from 0 to 50 rad/sec.

$$s_1 = -3 \qquad z_1 = e^{sT} = e^{(-3)(0.01)} = 0.9704$$

$$s_2 = -7 \qquad z_2 = e^{-0.07} = 0.9324$$

$$s_3 = -20 \qquad z_3 = e^{-0.2} = 0.8187$$

meaning

$$G(z) = \left( \frac{k}{(z-0.9704)(z-0.9324)(z-0.8187)} \right)$$

The DC gain of  $G(s)$  is

$$\left( \frac{1000}{(s+3)(s+7)(s+20)} \right)_{s=0} = 2.3810$$

Pick 'k' so that  $G(z)$  has the same DC gain

$$\left( \frac{k}{(z-0.9704)(z-0.9324)(z-0.8187)} \right)_{z=1} = 2.3810$$

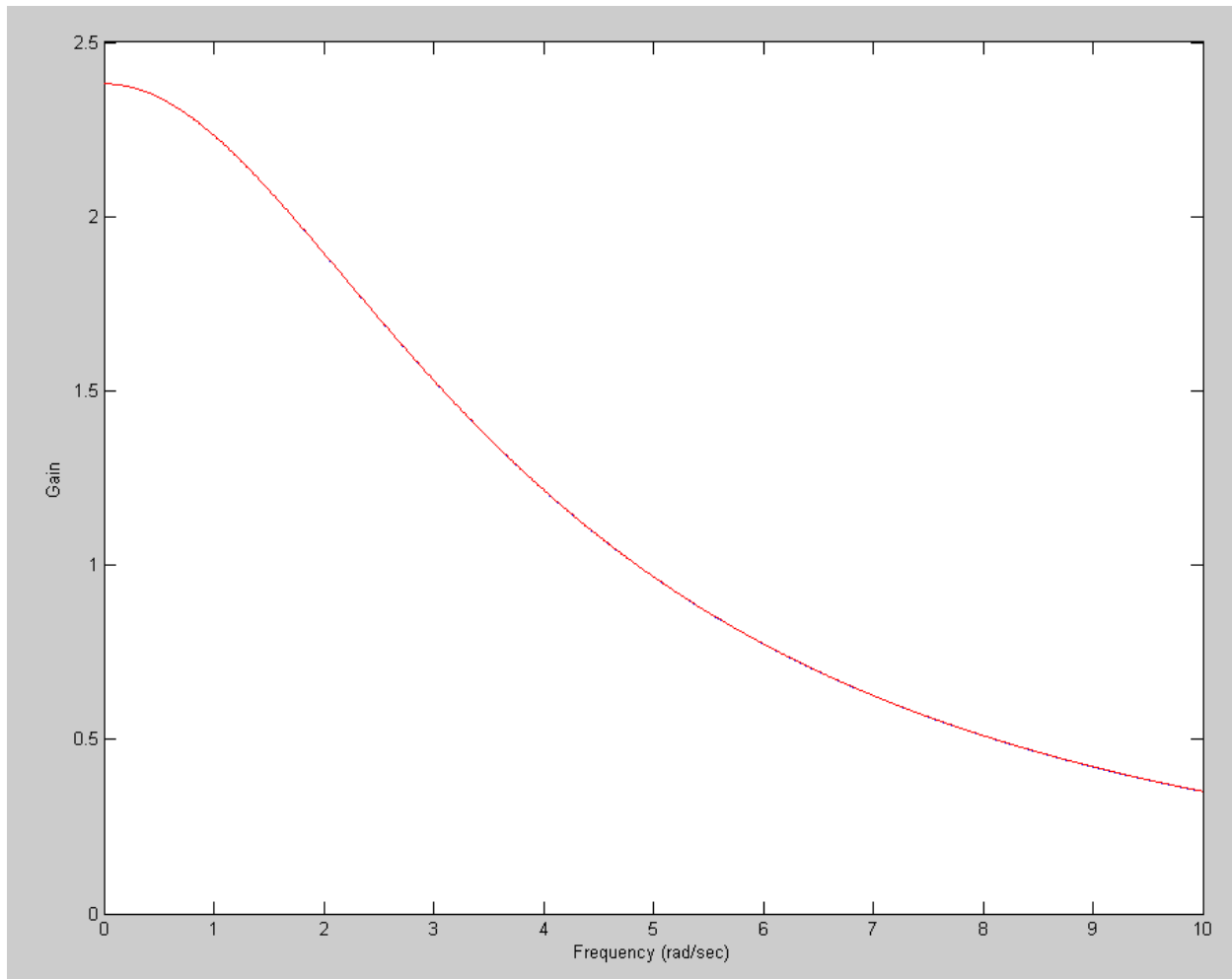
$$k = 0.00086235$$

so...

$$G(z) = \left( \frac{0.00086235}{(z-0.9704)(z-0.9324)(z-0.8187)} \right)$$

## Plotting the gain vs. frequency

```
>> w = [0:0.01:10]';  
>> s = j*w;  
>> Gs = 1000 ./ ( (s+3).*(s+7).*(s+20) );  
  
>> T = 0.01;  
>> z = exp(s*T);  
>> Gz = 0.00086235 ./ ( (z-0.9704).*(z-0.9324).*(z-0.8187) );  
  
>> plot(w,abs(Gs),'b',w,abs(Gz),'r');
```



$G(s)$  (blue) and  $G(z)$  (red).

Problem 4) Assume  $G(s)$  is the following band-pass filter:

$$G(s) = \left( \frac{6s}{(s+3+j20)(s+3-j20)} \right)$$

Design a digital filter,  $G(z)$ , which has approximately the same gain vs. frequency as  $G(s)$ . Assume a sampling rate of  $T = 0.01$  second.

Plot the gain vs. frequency for both filters from 0 to 50 rad/sec.

$$\begin{array}{ll} s_1 = 0 & z_1 = e^{sT} = e^0 = 1 \\ s_2 = -3 + j20 & z_2 = e^{sT} = 0.9511 + j0.1928 \\ s_3 = -3 - j20 & z_3 = e^{sT} = 0.9511 - j0.1928 \end{array}$$

So,  $G(z)$  looks like

$$G(z) = \left( \frac{k(z-1)}{(z-0.9511+j0.1928)(z-0.9511-j0.1928)} \right)$$

To find 'k', match the gain somewhere. DC doesn't work since the DC gain is zero. Pick some other frequency, such as 20 rad/sec

$$\left| \left( \frac{6s}{(s+3+j20)(s+3-j20)} \right)_{s=j20} \right| = 0.9972$$

Pick 'k' so that the gain matches

$$\begin{aligned} z &= e^{sT} = e^{j0.2} \\ \left| \left( \frac{k(z-1)}{(z-0.9511+j0.1928)(z-0.9511-j0.1928)} \right)_{z=e^{j0.2}} \right| &= 0.9972 \\ k &= 0.05794 \end{aligned}$$

meaning

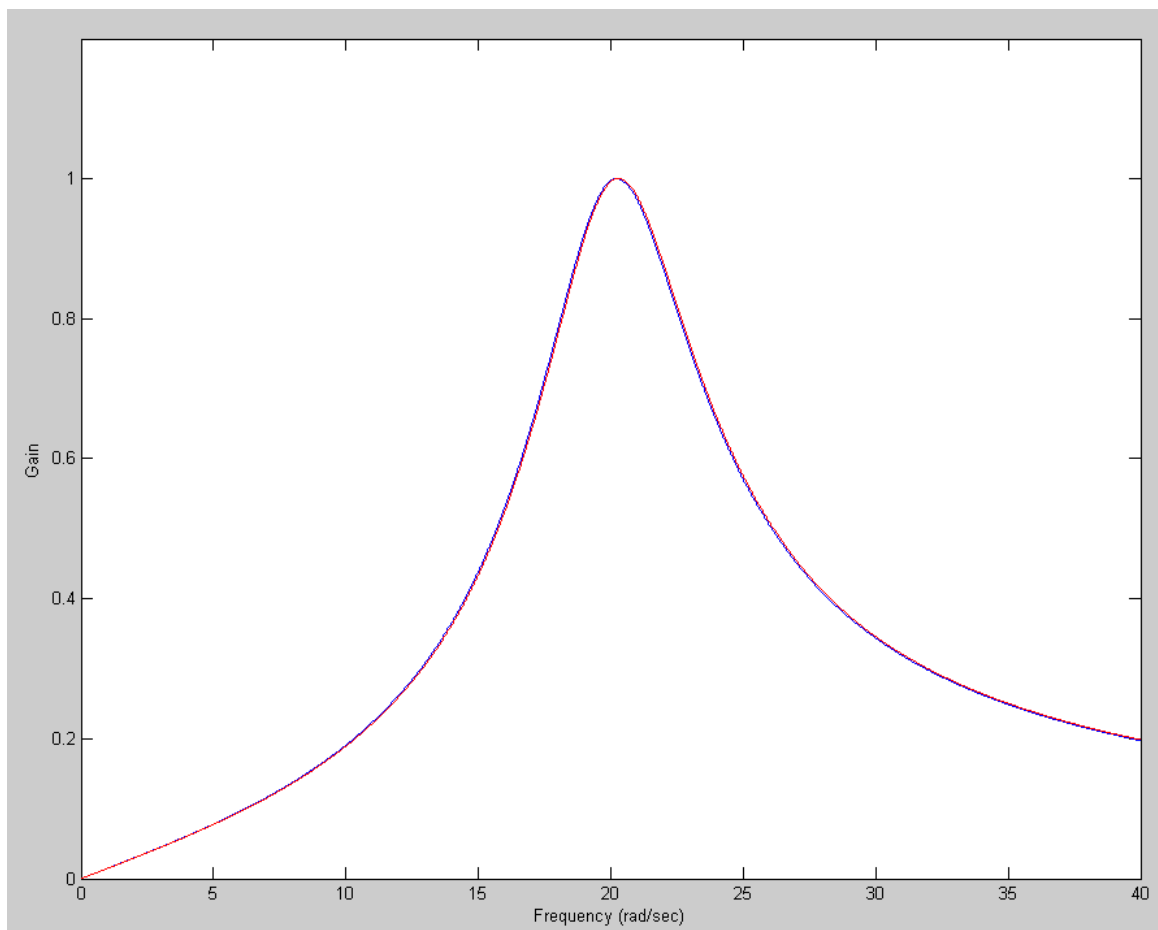
$$G(z) = \left( \frac{0.05794(z-1)}{(z-0.9511+j0.1928)(z-0.9511-j0.1928)} \right)$$

or

$$G(z) = \left( \frac{0.05794(z-1)}{z^2 - 1.902z + 0.9418} \right)$$

Plotting the gain vs. frequency for the two:

```
>> w = [0:0.01:40]';  
>> s = j*w;  
>> Gs = 6*s ./ ( (s+3+j*20).*(s+3-j*20) );  
>>  
>> T = 0.01;  
>> z = exp(s*T);  
>> Gz = 0.05794*(z-1) ./ ( z.^2 - 1.902*z + 0.9418 );  
  
>> plot(w,abs(Gs),'b',w,abs(Gz),'r');  
>> xlabel('Frequency (rad/sec)')  
>> ylabel('Gain');
```



G(s) (red) & G(z) (blue)

Problem 5) Write a C program to implement the digital filter,  $G(z)$

$$\frac{0.05794 (z-1)}{(z^2 - 1.902z + 0.9418)}$$

$$Y = \left( \frac{0.05794(z-1)}{z^2 - 1.902z + 0.9418} \right) X$$

Rewrite as

$$(z^2 - 1.902z + 0.9418)Y = 0.05794(z - 1)X$$

$$z^2 Y = (1.902z - 0.9418)Y + 0.05794(z - 1)X$$

$$y(k+2) = 1.902 y(k+1) - 0.9418 y(k) + 0.05794 (x(k+1) - x(k))$$

Shift time by 2 (change of variable)

$$y(k) = 1.902 y(k-1) - 0.9418 y(k-2) + 0.05794 (x(k-1) - x(k-2))$$

That's essentially the C program:

```
while(1) {  
  
    x2 = x1;  
    x1 = x0;  
    x0 = A2D_Read();  
  
    y2 = y1;  
    y1 = y0;  
    y0 = 1.902*y1 - 0.9418*y2 + 0.05794*( x1 - x2 );  
  
    D2A(y0);  
  
    Wait_10ms();  
  
}
```