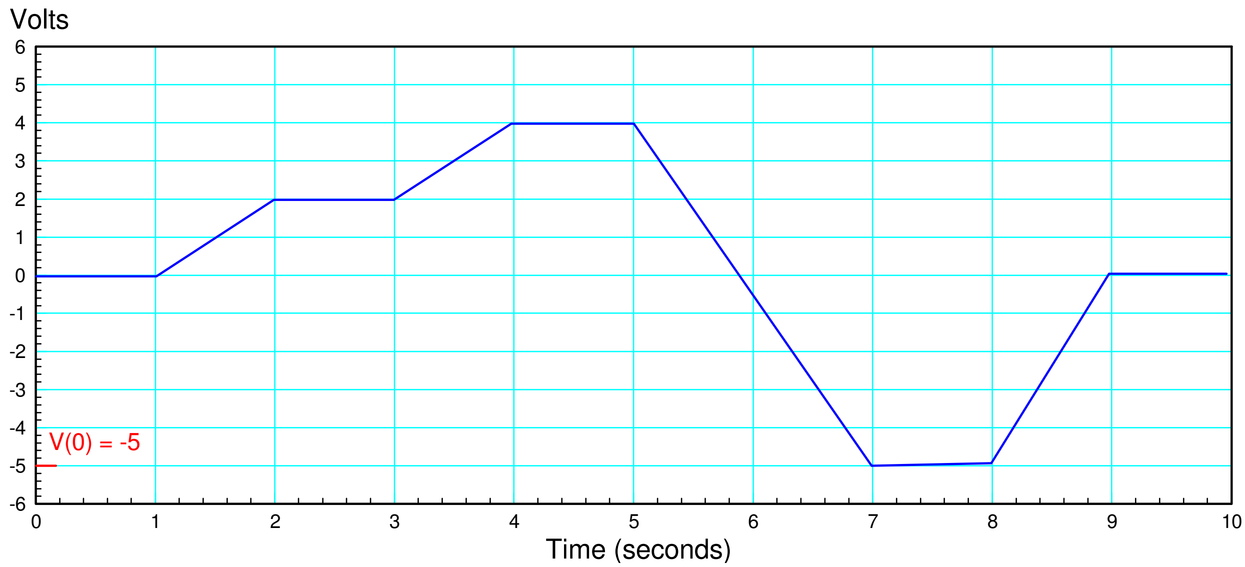
# ECE 111 - Homework #10

ECE 311 Circuits II - Capacitors & the Heat Equation

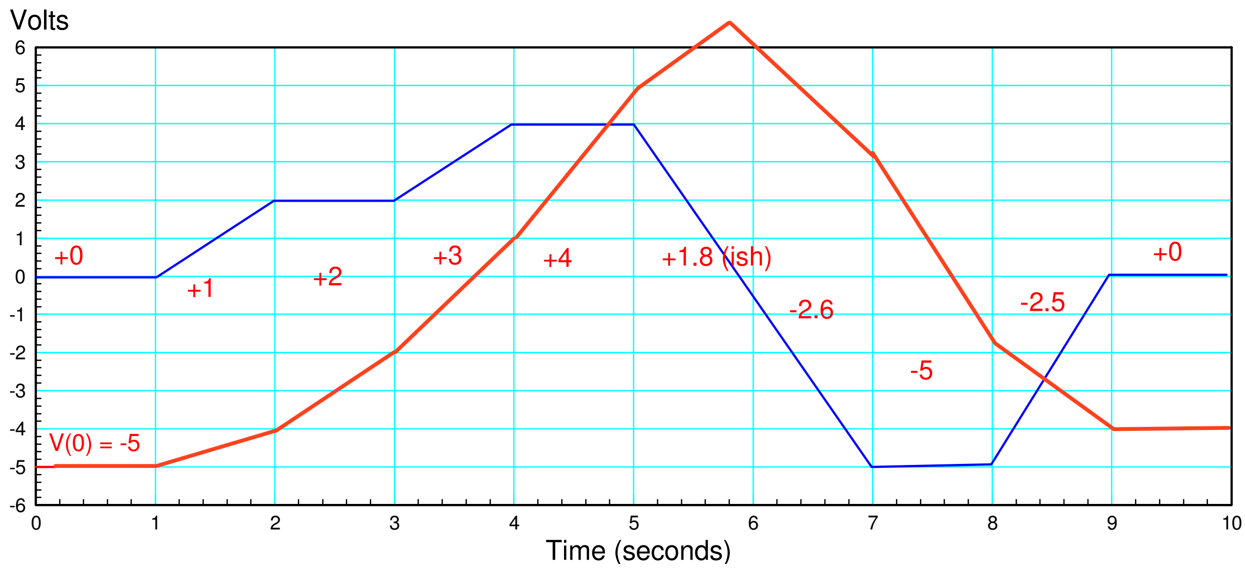
1) Assume the current flowing through a one Farad capacitor is shown below. Sketch the voltage. Assume V(0) = 0. The voltage is the integral of the current (capacitors are integrators)

$$V = \frac{1}{C} \int I \cdot dt$$

Assume the initial voltage on the capacitor is -5V.



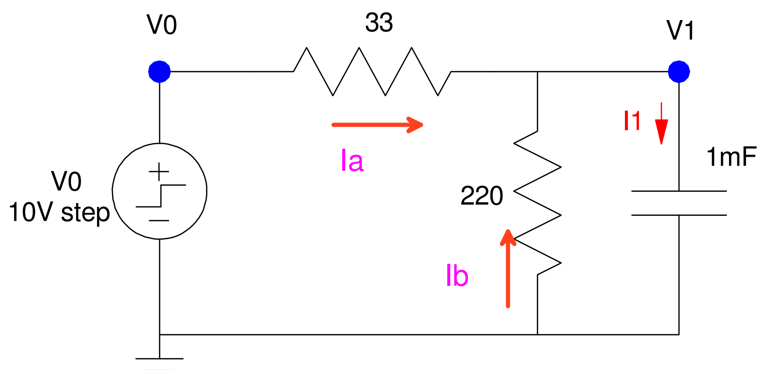

The integral is the area under the curve (positive goes up, negative goes down)



**1-Stage RC filter:**

2) Write the differential equation that describe this circuit. Note:

$$I_1 = C\frac{dV_1}{dt} = \sum(\text{current to node}V_1)$$




$$I_1 = C\frac{dV_1}{dt} = I_a + I_b$$

$$I_1 = 0.001V_1' = \left(\frac{V_0 - V_1}{33}\right) + \left(\frac{0 - V_1}{220}\right)$$

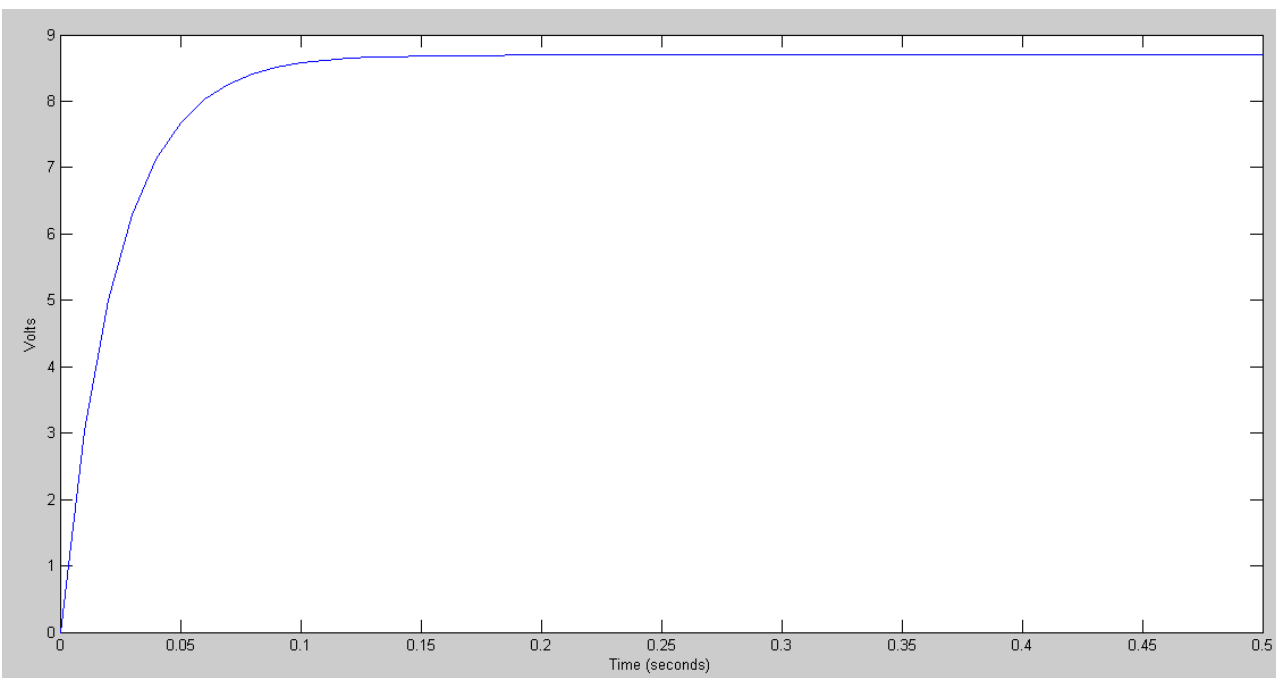
Doing some algebra

$$\boxed{V_1' = 30.303V_0 - 34.848V_1}$$

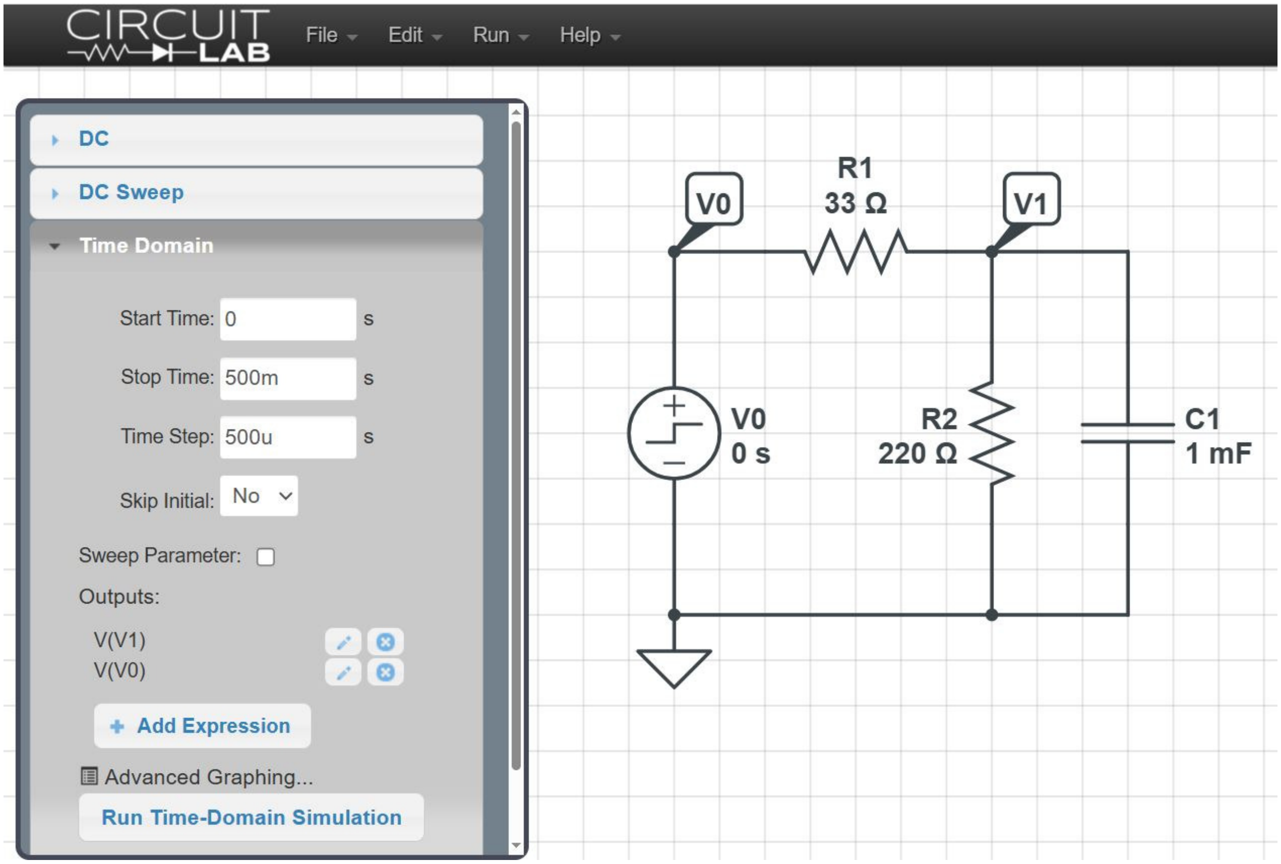
3) Find and plot V1(t) for 1/2 second using Matlab.

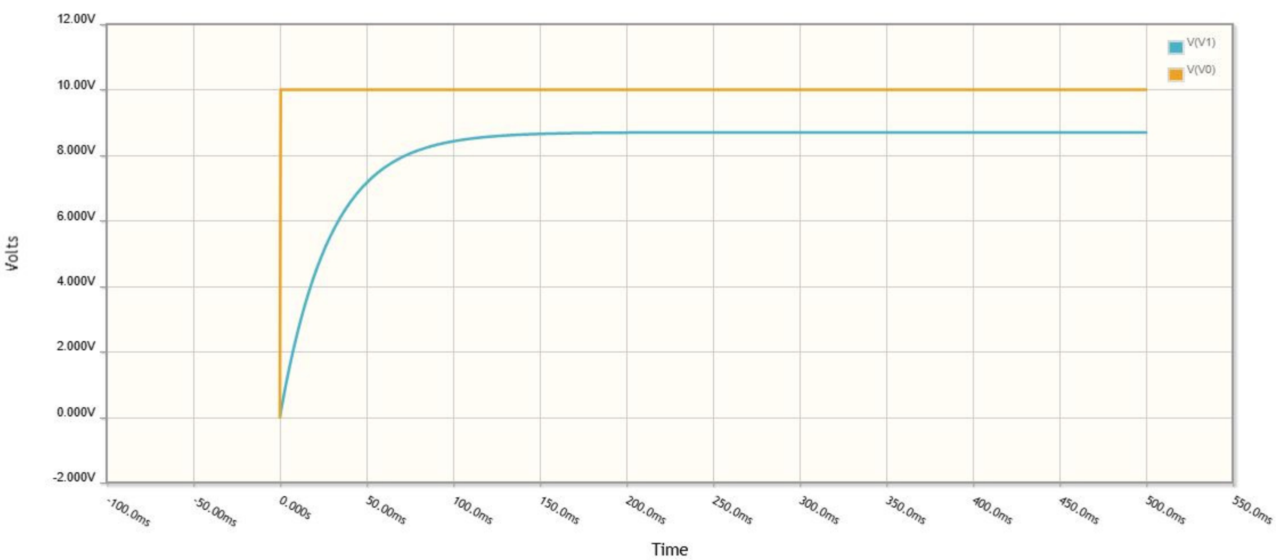
Matlab Code:

```
V0 = 10;
V1 = 0;
dt = 0.01;
t = 0;

y = [V1];

while(t < 0.5)

    dV1 = 30.303*V0 - 34.848*V1;

    V1 = V1 + dV1*dt;
    t = t + dt;

    y = [y ; V1];
end

t = [0:length(y)-1]' * dt;

plot(t,y);
xlabel('Time (seconds)');
ylabel('Volts');
```



4) Find and plot V1(t) for 1/2 second using CircuitLab

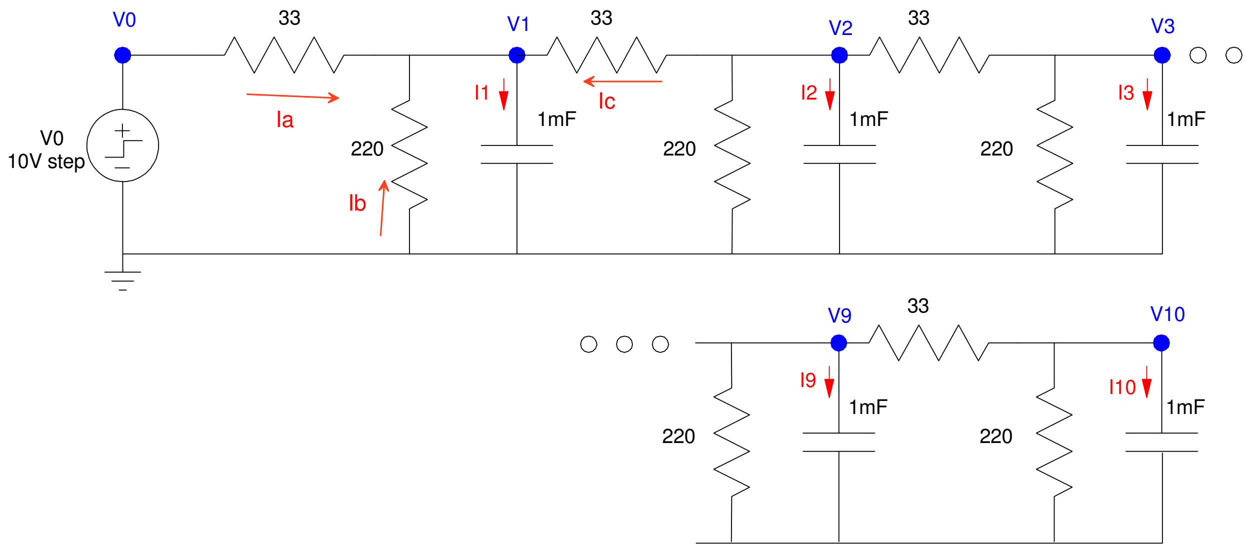


## 10-Stage RC Filter

5) Write the dynamics for a 10-stage RC filter (repeat for ten stages) as ten coupled differential equations:

$$I_1 = C\frac{dV_1}{dt} = \sum(\text{current to node} V_1)$$




At node V1:

$$I_1 = C\frac{dV_1}{dt} = I_a + I_b + I_c$$

$$I_1 = 0.001V_1' = \left(\frac{V_0 - V_1}{33}\right) + \left(\frac{0 - V_1}{220}\right) + \left(\frac{V_2 - V_1}{33}\right)$$

Doing some algebra

$$V_1' = 30.3030V_0 - 65.1515V_1 + 30.3030V_2$$

By symmetry, nodes 2..9 are similar

$$V_2' = 30.3030V_1 - 65.1515V_2 + 30.3030V_3$$

$$\vdots$$

$$V_9' = 30.3030V_8 - 65.1515V_9 + 30.3030V_{10}$$

Node #10 is slightly different since there is no node #11

$$I_{10} = 0.001V_{10}' = \left(\frac{V_9 - V_{10}}{33}\right) + \left(\frac{0 - V_{10}}{220}\right)$$

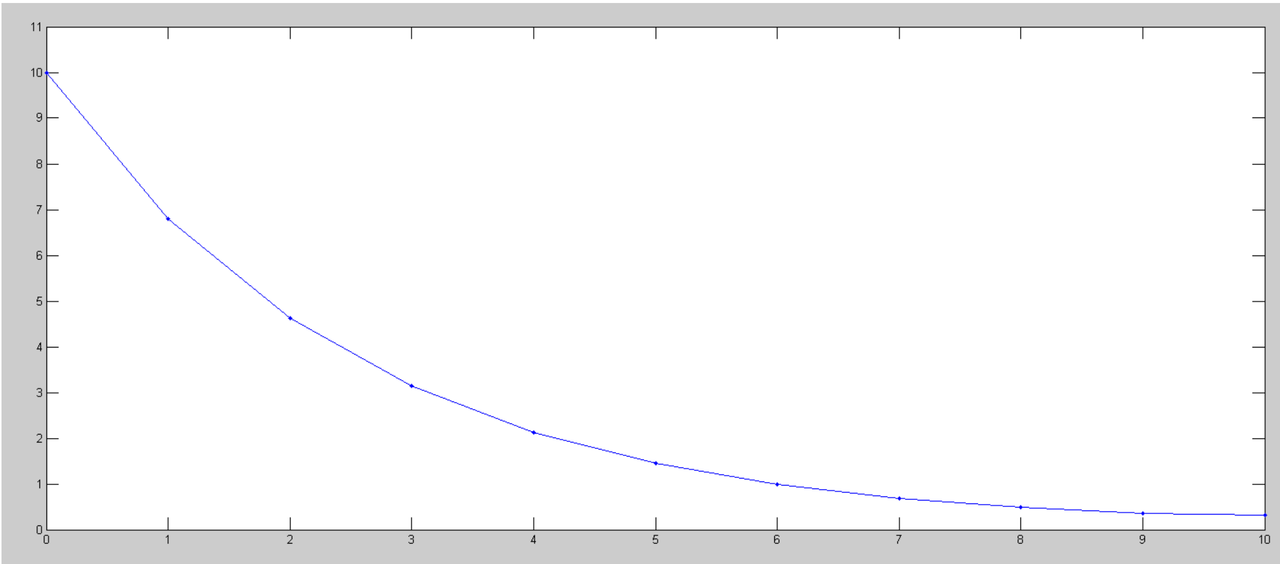
$$V_{10}' = 30.3030V_9 - 34.8484V_{10}$$

### Forced Response for a 10-Node RC Filter (heat.m):

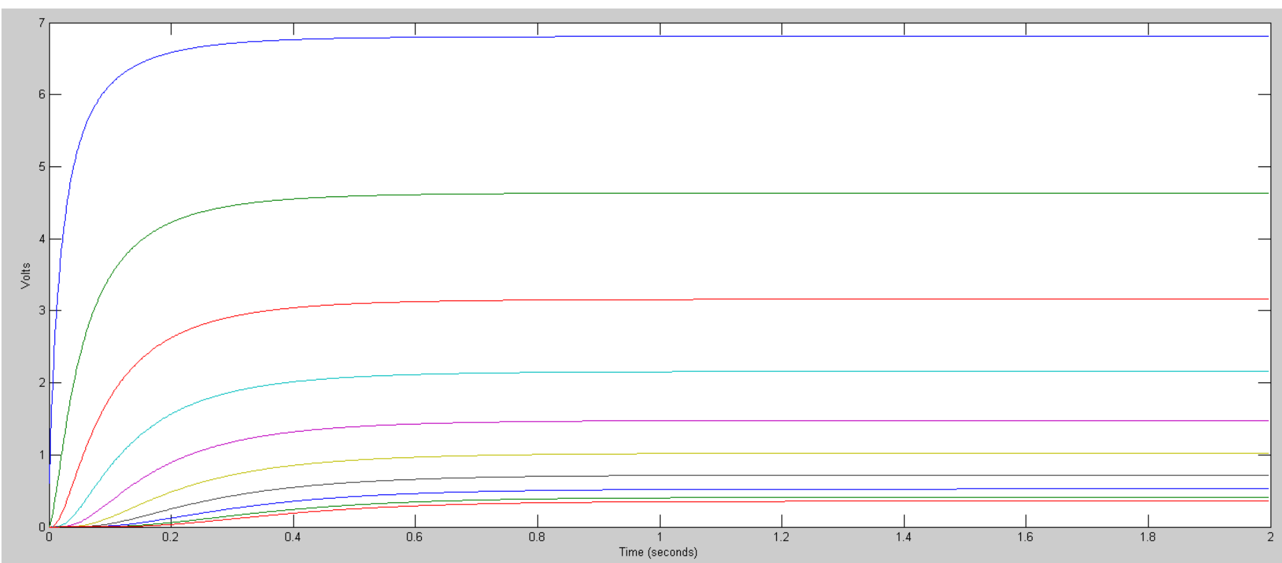
6) Using Matlab, solve these ten differential equations for $0 < t < 2$ s assuming

- The initial voltages are zero, and
- V0 = 10V.

Resulting voltage vs. node number




Resulting voltages vs. time



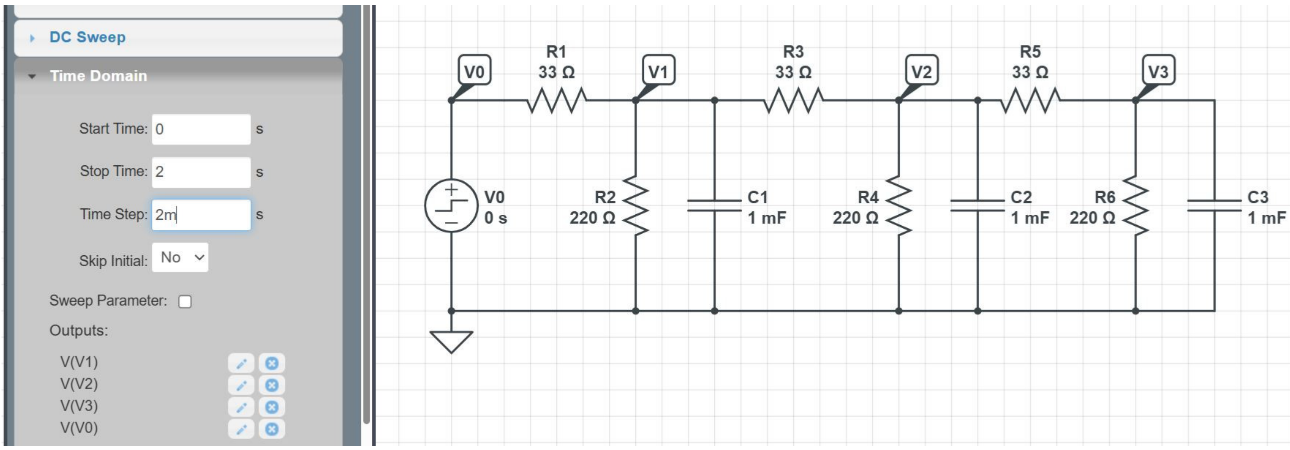
Matlab Code:

```
% 10-stage RC Filter
V = zeros(10,1);
dV = zeros(10,1);
V0 = 10;
dt = 0.002;
t = 0;

y = [];

while(t < 2)

   dV(1)  = 30.3030*V0   - 65.1515*V(1) + 30.3030*V(2);
   dV(2)  = 30.3030*V(1) - 65.1515*V(2) + 30.3030*V(3);
   dV(3)  = 30.3030*V(2) - 65.1515*V(3) + 30.3030*V(4);
   dV(4)  = 30.3030*V(3) - 65.1515*V(4) + 30.3030*V(5);
   dV(5)  = 30.3030*V(4) - 65.1515*V(5) + 30.3030*V(6);
   dV(6)  = 30.3030*V(5) - 65.1515*V(6) + 30.3030*V(7);
   dV(7)  = 30.3030*V(6) - 65.1515*V(7) + 30.3030*V(8);
   dV(8)  = 30.3030*V(7) - 65.1515*V(8) + 30.3030*V(9);
   dV(9)  = 30.3030*V(8) - 65.1515*V(9) + 30.3030*V(10);
   dV(10) = 30.3030*V(9) - 34.8484*V(10);

   V = V +  dV*dt;
   t = t + dt;

   y = [y ; V'];

   plot([0:10], [V0;V], '.-');
   ylim([0,11]);
   pause(0.01);

   end

t = [0:length(y)-1]' * dt;
plot(t,y);
xlabel('Time (seconds)');
ylabel('Volts');
```

7) Using CircuitLab, find the response of this circuit to a 10V step input. *note: It's OK if you only build this circuit to 3 nodes...*

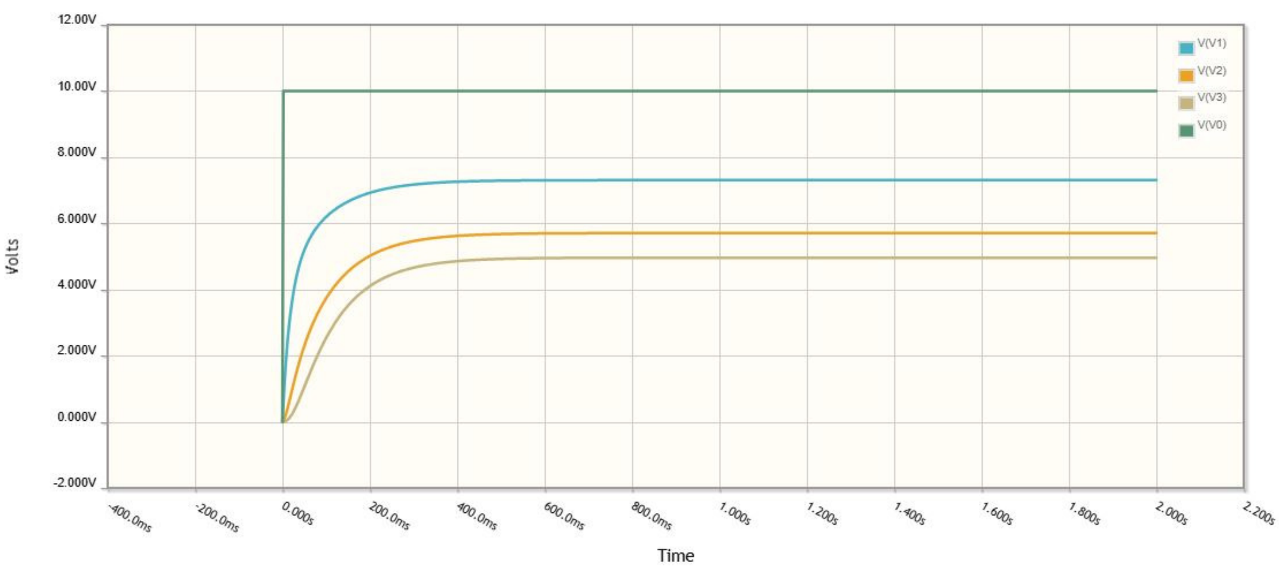
note: The result is about the same (a little different since only three nodes are simulated)

CiruitLab circuit




Resulting volages:

- same as computed in Matlab
- CircuitLab uses numerical integration to find the voltages vs. time, same as we did in Matlab



## Natural Response: Eigenvectors and Eigenvalues

8) Assume V0 = 0V. Determine the initial conditions of V1..V10 so that

- The maximum voltage is 10V and
- 5a) The voltages go to zero as slow as possible
- 5b) The voltages go to zero as fast as possible.

Simulate the response for these initial conditions in Matlab.

This is an eigenvalue & eigenvector problem. The eigenvectors and eigenvales can be found using Matlab

First, input the dynamics in matrix form:

$$\frac{dV}{dt} = AV$$

```
>> A = zeros(10,10);
>> for i=1:9
A(i,i) = -65.1515;
A(i+1,i) = 30.3030;
A(i,i+1) = 30.3030;
end
>> A(10,10) = -34.8484;
```

```
 -65.1515   30.3030         0         0         0         0         0         0         0         0
  30.3030  -65.1515   30.3030         0         0         0         0         0         0         0
        0   30.3030  -65.1515   30.3030         0         0         0         0         0         0
        0         0   30.3030  -65.1515   30.3030         0         0         0         0         0
        0         0         0   30.3030  -65.1515   30.3030         0         0         0         0
        0         0         0         0   30.3030  -65.1515   30.3030         0         0         0
        0         0         0         0         0   30.3030  -65.1515   30.3030         0         0
        0         0         0         0         0         0   30.3030  -65.1515   30.3030         0
        0         0         0         0         0         0         0   30.3030  -65.1515   30.3030
        0         0         0         0         0         0         0         0   30.3030  -34.8484
```

Find the eigenvalues and eigenvectors using Matlab

```
>> [M,V] = eig(A)

M =

 fast mode                                                                                      slow mode
 -0.1286   -0.2459    0.3412    0.4063    0.4352    0.4255    0.3780    0.2969   -0.1894    0.0650
  0.2459    0.4063   -0.4255   -0.2969   -0.0650    0.1894    0.3780    0.4352   -0.3412    0.1286
 -0.3412   -0.4255    0.1894   -0.1894   -0.4255   -0.3412    0.0000    0.3412   -0.4255    0.1894
  0.4063    0.2969    0.1894    0.4352    0.1286   -0.3412   -0.3780    0.0650   -0.4255    0.2459
 -0.4352   -0.0650   -0.4255   -0.1286    0.4063    0.1894   -0.3780   -0.2459   -0.3412    0.2969
  0.4255   -0.1894    0.3412   -0.3412   -0.1894    0.4255   -0.0000   -0.4255   -0.1894    0.3412
 -0.3780    0.3780    0.0000    0.3780   -0.3780    0.0000    0.3780   -0.3780   -0.0000    0.3780
  0.2969   -0.4352   -0.3412    0.0650    0.2459   -0.4255    0.3780   -0.1286    0.1894    0.4063
 -0.1894    0.3412    0.4255   -0.4255    0.3412   -0.1894    0.0000    0.1894    0.3412    0.4255
  0.0650   -0.1286   -0.1894    0.2459   -0.2969    0.3412   -0.3780    0.4063    0.4255    0.4352

V =
 -123.0649 -115.2265 -102.9387  -87.2934  -69.6806  -51.6654  -34.8485  -20.7241  -10.5474   -5.2224
```

The eigenvector is *how* the energy decays

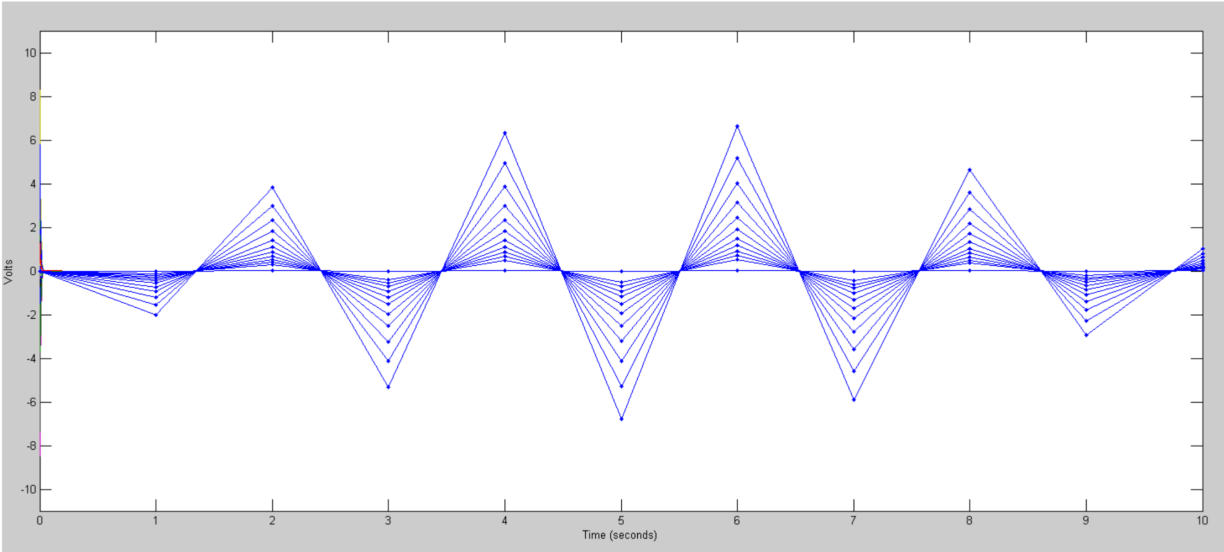
- Something decays slowly as exp(-5.22t) (the eigenvector in red)
- Something decays quickly as exp(-123.06t) (the eigenvector in blue)

To simulate the fast eigenvector

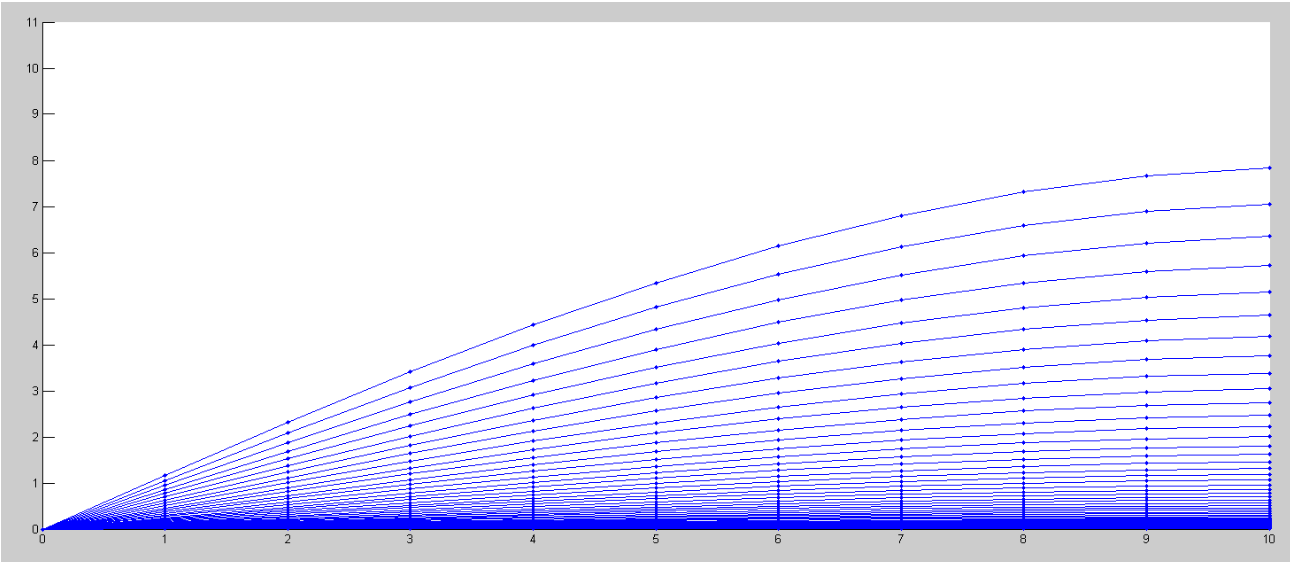
- Make the forcing funciton zero (V0 = 0)
- Make the initial condition the fast eigenvector
- Let the simulation run

The energy decays as exp(-123t) (very fast)

```
V = M(:,1);
dV = zeros(10,1);
V0 = 0;
dt = 0.002;
t = 0;
```




Fast Mode: Voltages plotted every 2ms



Slow Mode: Voltages plotted every 20ms

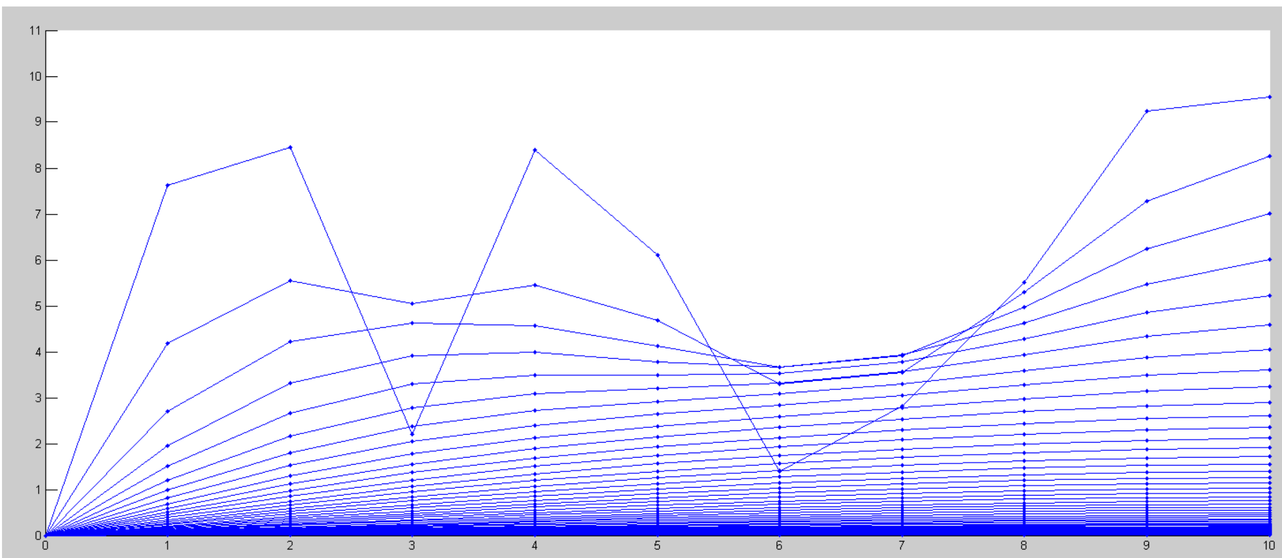
9) Assume Vin = 0V. Pick random voltages for V1 .. V10 in the range of (0V, 10V):

```
V = 10 * rand(10,1)
```

Plot the votlages at t = 2. Which eigenvector does it look like?

With a random initial condition, all 10 eigenvectors are excited

- The fast modes decay quickly,
- Leaving the slow mode




Random Initial COndition: Voltages plotted every 20ms